Fujisaki, M.
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CONTROL OF DIFFUSION PROCESSES IN R^d AND BELLMAN EQUATION WITH DEGENERATION

MASATOSHI FUJISAKI

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0. Introduction

In this paper we consider the existence and uniqueness of solutions of the following Bellman equation:

$$(0.1)\quad \begin{cases} \inf_{\alpha\in A}\{\partial v/\partial s+1/2\sum_{1\leq i,j\leq \nu} a_{ij}(\alpha, s, x)\partial^2 v/\partial x_i\partial x_j+ \\ \sum_{i=1}^{d} b_i(\alpha, s, x)\partial v/\partial x_i-c(\alpha, s, x)v+L(\alpha, s, x)\} = 0\,, \\ v(T, x) = h(x) \end{cases}$$

where $1\leqq\nu<d$, A is a separable metric space and (a_{ij}), $1\leqq i, j\leqq\nu$, is a positive definite matrix.

W.H. Fleming already considered in [1] the following equation which is more restrictive than Eq. (0.1):

$$(0.2)\quad \begin{cases} \partial v/\partial s+1/2\sum_{1\leq i,j\leq \nu} a_{ij}(s, x)\partial^2 v/\partial x_i\partial x_j+\sum_{i=\nu+1}^{d} b_i(s, x)\partial v/\partial x_i \\ +\inf_{\alpha\in A}\{\sum_{i=1}^{\nu} b_i(\alpha, s, x)\partial v/\partial x_i-c(\alpha, s, x)v+L(\alpha, s, x)\} = 0\,, \\ v(T, x) = h(x)\,. \end{cases}$$

In [1] he also considered the deterministic case that $\nu=0$ in Eq. (0.2). His approach to this equation is as follows; consider stochastic control problem for a system described by the following stochastic differential equation:

$$(0.3)\quad dX_t = b(\alpha_t, t, X_t)dt+\sigma(t, X_t)dB_t,\ \ X_s = x\,,$$

where $b_i(\alpha, s, x)=b_i(s, x)$ for all $i=\nu+1, \cdots, d$, $\sigma=\begin{pmatrix}\bar\sigma & 0\\ 0 & 0\end{pmatrix}$, $\bar\sigma$ is a nonsingular (ν, ν)-matrix, (B_t) is a vector valued Brownian motion, x is a vector of R^d and (α_t) is a non-anticipative control variable having values in A. Define the cost v by the following formula:

$$(0.4)\quad v(s, x) = \inf E[\int_s^T L(\alpha_t, t, X_t^{\alpha,s,x})\exp\{-\int_s^t c(\alpha_r, r, X_r^{\alpha,s,x})dr\}\,dt \\ +h(X_T^{\alpha,s,x})\exp\{-\int_s^T c(\alpha_t, t, X_t^{\alpha,s,x})dt\}]\,,$$