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Osaka J. Math.
22 (1985), 815–819

ON PROJECTIVE MODULES OVER DIRECTLY FINITE REGULAR RINGS SATISFYING THE COMPARABILITY AXIOM

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(Received November 6, 1984)

In [2], J. Kado has studied simple directly finite regular rings satisfying the comparability axiom, and completely determined the directly finiteness of projective modules over these rings. In this paper, without the assumption of simplicity in [2], we shall study projective modules over directly finite regular rings satisfying the comparability axiom. In Theorem 6, we shall give a criterion of the directly finiteness over these rings. Using this criterion, in Theorem 7, we shall show the following result: Let R be a directly finite regular ring satisfying the comparability axiom. If P and Q are directly finite projective R-modules, then so is $P\oplus Q$.

Throughout this paper, R is a ring with identity and R-modules are unitary right R-modules. If M and N are R-modules, then the notation $N\lesssim M$ (resp. $N\lesssim\oplus M$) means that N is isomorphic to a submodule of M (resp. N is isomorphic to a direct summand of M). For a cardinal number α and an R-module M, αM denotes a direct sum of α-copies of M.

First we recall some definitions and well-known results (cf. [1]).

DEFINITION. A ring R is *directly finite* if $xy=1$ implies $yx=1$, for all $x, y\in R$. An R-module M is *directly finite* if $End_R(M)$ is directly finite. A ring R (a module M) is *directly infinite* if it is not directly finite. It is well-known that M is directly finite if and only if M is not isomorphic to a proper direct summand of M itself. A regular ring R is said to *satisfy the comparability axiom* provided that, for any $x, y\in R$, either $xR\lesssim yR$ or $yR\lesssim xR$, or equivalently, for any finitely generated projective R-modules P and Q, either $P\lesssim Q$ or $Q\lesssim P$. A ring R is said to be *unit-regular* if, for each $x\in R$, there is a unit (i.e. an invertible element) u of R such that $xux=x$.

Lemma 1. (a) *Every directly finite regular ring satisfying the comparability axiom is unit-regular* (cf. [1, Theorem 8.12]).

(b) *Let R be a unit-regular ring. Then,*

(1) *Every finitely generated projective R-module is directly finite* ([1,