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Osaka J. Math.
22 (1985), 811-814

A COMMUTATIVITY THEOREM FOR RINGS. II

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(Received October 22, 1984)

Throughout the present paper, R will represent a ring with center C, and D the commutator ideal of R. A ring R is called left (resp. right) s-unital if $x\in Rx$ (resp. $x\in xR$) for every $x\in R$; R is called s-unital if R is both left and right s-unital. Given a positive integer n, we say that R has the property $Q(n)$ if for any $x, y\in R$, $n[x, y]=0$ implies $[x, y]=0$ (see [1]).

Our present objective is to generalize [2, Theorem] for left s-unital rings as follows:

Theorem. *Let $n>0$, r, s and t be non-negative integers and let $f(X, Y)=\sum_{i=1}^{r}\sum_{j=2}^{s} f_{ij}(X, Y)$ be a polynomial in two noncommuting indeterminates X, Y with integer coefficients such that each f_{ij} is a homogeneous polynomial with degree i in X and degree j in Y and the sum of the coefficients of f_{ij} equals zero. Suppose a left s-unital ring R satisfies the polynomial identity*

$$X^t[X^n, Y]-f(X, Y) = 0\,. \tag{1}$$

If either $n=1$ or $r=1$ and R has the property $Q(n)$, then R is commutative.

We shall use freely the following well known result stated without proof.

Lemma. *Let x, y be elements of a ring with 1, and let k be a positive integer. If $x^k y=0=(x+1)^k y$ then $y=0$.*

Proof of Theorem. Let y be an arbitrary element of R, and choose an element e of R such that $ey=y$. Then (1) gives $y-ye^n=f(e, y)\in yR$. We have thus seen that R is right s-unital, and hence s-unital. Therefore, in view of [1, Proposition 1], it suffices to prove the theorem for R with 1.

Observe that D is a nil ideal of R, by a theorem of Kezlan-Bell (see, e.g., [1, Proposition 2]), since $x=e_{11}$ and $y=e_{12}$ fail to satisfy (1).

I) We consider first the case $n=1$. Let a, b be elements of R. By Lemma, it is easy to see that if $x^t a[x, b]=0$ for all $x\in R$ then $a[x, b]=0$. Noting this fact, we can apply the argument employed in the proof of [2, Theorem] to see the commutativity of R.