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EFFECTIVE LOWER BOUNDS ON LARGE FUNDAMENTAL UNITS OF REAL QUADRATIC FIELDS

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There is considerable interest in how large the fundamental units of real quadratic fields may be. For example, when factoring a rational integer using the Continued Fraction Method, see [2], one avoids expansions of quadratic surds in fields with too small a fundamental unit. More classically Gauss' conjecture that infinitely many real quadratic fields have class number one could be shown if fields with huge enough fundamental units were known.

In 1971 Yamamoto [4] gave classes of large fundamental units. It is well known that in real quadratic fields the class number and the logarithm of the fundamental unit are roughly inversely proportional. See Hua [1, p. 336] for a precise statement. Yamamoto's theorem, and ours, uses hypotheses about ideals being principal to imply lower bounds on fundamental units.

Let $\boldsymbol{N}$ denote the natural numbers and $\boldsymbol{Q}$ the rational numbers. For $d\in\boldsymbol{N}$ with $d>1$ and d not a square we write $\boldsymbol{Q}(\sqrt{d})$ for the quadratic field with discriminant $D=D_d$, fundamental unit $\varepsilon=\varepsilon_d$, class number $h=h_d$, and ring of integers $\mathcal{O}_d$. Yamamoto's theorem is:

Theorem 1.1 (Yamamoto). *Let $p_1, \cdots, p_n\in\boldsymbol{N}$ be primes. Let I be a set of infinitely many square-free positive integers. Suppose that for all $1\leq i\leq n$ and $d\in I$ that one has $(p_i)=\mathcal{P}_i\bar{\mathcal{P}}_i$, $gcd(p_i, d)=1$ and $\mathcal{P}_i$ principal in $\mathcal{O}_d$, then there is a constant c so that*

$$\log\varepsilon_d>c(\log d)^{n+1}\quad \textit{for } d\in I\,.$$

Yamamoto then gave a class of sets I with $n=2$. In this paper we will generalize his theorem so that it is effective. Since our theorem is not asymptotic there is no need for an infinite set I. The hypothesis that $p_1, \cdots, p_n$ be prime will be weakened. Furthermore, we will give many other classes of examples with $n=2$. That is, we will give many fields whose fundamental units are large in this sense. It should be remarked that these theorems fall far short of settling Gauss' conjecture.

1. Effective lower bounds on fundamental units

We give here several Lemmas that we will need. See Yamamoto [4]