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A CLASS OF TRANSCENDENTAL FUNCTIONS CONTAINING ELEMENTARY AND ELLIPTIC ONES

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0. Introduction

In this paper we shall present a differential extension field which is wider than Liouville's one and contains elliptic functions. The irreducibility of ordinary differential operators over our field will be investigated.

Liouville proved in [6] that if a linear homogeneous differential equation of the second order over the rational function field $\boldsymbol{C}(x)$ admits a non-trivial solution which is liouvillian over $\boldsymbol{C}(x)$ then it admits a non-trivial solution whose logarithmic derivative is algebraic over $\boldsymbol{C}(x)$ (cf. Ritt [11, chapter 4]). In his [13], Rosenlicht extended this result to the case of general order. As was mentioned there, the theorem of his can be obtained through Picard-Vessiot theory (confer with Kolchin [3]). We shall further extend this.

In [14], Siegel proved a similar theorem. That is to say, if a linear homogeneous differential equation of the second order over $\boldsymbol{C}(x)$ admits a non-trivial solution which satisfies an algebraic differential equation over $\boldsymbol{C}(x)$ then it admits a non-trivial solution whose logarithmic derivative is algebraic over $\boldsymbol{C}(x)$. This result was generalized by Goldman [1] in the case of general order, and further by Singer [15] in the non homogeneous case. Their methods depend upon respectively the Low Power Theorem of Ritt and the valuation theory. The latter was utilized effectually first by Rosenlicht [Publ. Math. Inst. HES., 36 (1969), 15–22]. Another generalization was established by Oleinikov [9]: Let F be a differential field consisting of meromorphic functions in some domain. If a linear homogeneous differential equation of order n over F admits a non-trivial solution which satisfies an algebraic differential equation over F of order less than n, then it admits a non-trivial solution which satisfies a homogeneous differential equation over F of order less than n. His method is aalytical. We shall give a differential-algebraic proof of this theorem through considering formal infinite series in an arbitrary constant (cf. Ritt [12, chapter 3]).

Let K be an ordinary differential field of characteristic 0 with a differentiation D. Throughout this paper we fix a universal differential field extension