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Osaka J. Math.
22 (1985), 463–488

CHERN CHARACTERS ON COMPACT LIE GROUPS OF LOW RANK

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

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(Received August 8, 1984)

0. Introduction

Let G be a compact, simply connected, simple Lie group of rank l. G has l irreducible representations $\rho_1, \cdots, \rho_l$, whose highest weights are the fundamental weights $\omega_1, \cdots, \omega_l$ respectively (see [19]). Then the representation ring $R(G)$ of G is a polynomial algebra $Z[\rho_1, \cdots, \rho_l]$. By the theorem of Hodgkin [16], the $Z/2$-graded K-theory $K^*(G)$ of G is an exterior algebra $\Lambda_Z(\beta(\rho_1), \cdots, \beta(\rho_l))$, where $\beta: R(G) \to K^*(G)$ is the map introduced in [16]. Therefore the Chern character $ch: K^*(G) \to H^*(G; Q)$ is injective [5]. We may write

$$H^*(G; Q) = \Lambda_Q(x_{2m_1-1}, x_{2m_2-1}, \cdots, x_{2m_l-1})$$

where $2=m_1 \leq m_2 \leq \cdots \leq m_l$ and $\deg x_{2m_j-1}=2m_j-1$. If each x_{2m_j-1} is chosen to be integral and not divisible by any other integral classes, we can assign to a representation $\lambda: G \to U(n)$ the rational numbers $a(\lambda, 1), \cdots, a(\lambda, l)$ by the equation

$$ch\beta(\lambda) = \sum_{j=1}^{l} a(\lambda, j) x_{2m_j-1}.$$

In view of [21] and [23], the $a(\lambda, j)$ are closely related to the *Dynkin coefficients* of λ [14]. On the other hand, as is noted by Atiyah [4, Proposition 1], the determinant of the $l \times l$ matrix $(a(\rho_i, j))$ is equal to 1. We remark that for any system of generators $\{\lambda_1, \cdots, \lambda_l\}$ of the ring $R(G)$, the determinant of $(a(\lambda_i, j))$ is also 1.

In this paper, with a suitable system of generators of $R(G)$, we shall describe the resulting matrix explicitly for the groups G with $l \leq 4$ without using the above informations. Indeed, we deal with the following cases:

$l=2$, $G=\mathrm{SU}(3)$, $\mathrm{Sp}(2)$, G_2.
$l=3$, $G=\mathrm{SU}(4)$, $\mathrm{Spin}(7)$, $\mathrm{Sp}(3)$.
$l=4$, $G=\mathrm{SU}(5)$, $\mathrm{Spin}(9)$, $\mathrm{Sp}(4)$, $\mathrm{Spin}(8)$, F_4.