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ON THE SPECTRUM REPRESENTING ALGEBRAIC K-THEORY FOR A FINITE FIELD

Dedicated to Professor Nobuo Shimada on his sixtieth birthday

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Let r be an odd prime power. Let $\boldsymbol{F}_r$ denote the field with r elements. According to [11] and others, there exists a (-1)-connected Ω-spectrum $\boldsymbol{KF}_r$ whose 0-th space is $\boldsymbol{Z}\times BGL\boldsymbol{F}_r^+$, where $BGL\boldsymbol{F}_r^+$ is the plus construction of the classifying space of $GL\boldsymbol{F}_r$. $\boldsymbol{KF}_r$ is a ring spectrum with a unit.

Let p be an odd prime. The object of this paper is the localization of $\boldsymbol{KF}_r$ at p, $\boldsymbol{KF}_{r(p)}$, for the case that r gives a generator of the group of units $(\boldsymbol{Z}/p^2)^\times$. Then the associated generalized cohomology theory $K\boldsymbol{F}_r^*(\ ;\boldsymbol{Z}_{(p)})$ appears as a secondary cohomology theory determined by a certain stable operation in connected complex K-theory localized at p. From this interpretation we deduce some results about the multiplicative structure on $\boldsymbol{KF}_{r(p)}$, which are basic to the study of the ring structure of $K\boldsymbol{F}_{r*}(CP^\infty;\boldsymbol{Z}_{(p)})$ etc. In particular we can characterize the product on $\boldsymbol{KF}_{r(p)}$ by a certain property.

For simplicity we write $\boldsymbol{A}$ for $\boldsymbol{KF}_{r(p)}$ (see [8]). We shall work in the homotopy category of CW-spectra (see [3, III]).

The paper is organized as follows. In § 0 we collect several results on $\boldsymbol{A}$. In § 1 we compute $H^*(\boldsymbol{A};\boldsymbol{Z}/p)$. In § 2 we compute $H_*(\boldsymbol{A};\boldsymbol{Z}/p)$. In § 3 we consider the left coaction of $\mathcal{A}_*$ on $H_*(\boldsymbol{A};\boldsymbol{Z}/p)$ and discuss the $\mathcal{B}$-module structure of $H^*(\boldsymbol{A};\boldsymbol{Z}/p)$, where $\mathcal{B}=\Lambda(Q_0, Q_1)\subset\mathcal{A}$. In § 4 we prove our main results, which are Theorems 4.3 and 4.5.

0. The spectrum A

Let p be a fixed odd prime. Let $\boldsymbol{bu}_{(p)}$ be the Ω-spectrum representing connected complex K-theory localized at p. This is a ring spectrum with a unit and $\pi_*(\boldsymbol{bu}_{(p)})=\boldsymbol{Z}_{(p)}[u]$ where $|u|=2$. It is known that

$$\boldsymbol{bu}_{(p)}=\bigvee_{j=1}^{p-1}\Sigma^{2(j-1)}\boldsymbol{G}$$

for a spectrum $\boldsymbol{G}$ [6]. This is a ring spectrum with a unit and $\pi_*(\boldsymbol{G})=\boldsymbol{Z}_{(p)}[v]$ where $|v|=2(p-1)$. According to [4], if $\kappa\colon \boldsymbol{G}\to\boldsymbol{bu}_{(p)}$ is the injection, then the diagram