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Osaka J. Math.
22 (1985), 433–446

THE SPLITTING OF FINITELY GENERATED MODULES

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(Received August 20, 1984)

Throughout this paper rings are commutative with identity and modules are unital. The purpose of this paper is to study splitting properties of modules with respect to their torsion submodule, with special attention to the cases of integral domains and (von Neumann) regular rings.

The notion of torsion theory has been used in the literature to define the splitting properties we are concerned with. Recall that a (hereditary) torsion theory over the ring R is defined by a class of R-modules which is closed under the formation of submodules, homomorphic images, extensions and direct sums; this class is called the torsion class of the theory. Given such a class, every R-module has a largest submodule in the class, called the torsion submodule, with torsionfree factor module. A torsion theory is said to have the *finitely generated splitting property* (FGSP) if the torsion submodule of every finitely generated module splits off as a direct summand of the module. A weaker condition (but more susceptible to full analysis) is the *cyclic splitting property* (CSP): the torsion submodule of every cyclic module splits off.

The FGSP question was largely inspired by several now classical theorems of I. Kaplansky [9], [10], which can be paraphrased as this single statement: the usual torsion theory for an integral domain R has FGSP if and only if R is a Prufer ring. One of the principal results of this paper suitably extends this result to arbitrary torsion theories over an integral domain. Of course, Kaplansky's result follows as a special case. As a first step we analyze CSP over domains. In fact, FGSP over domains reduces to CSP plus an arithmetic property. We also examine CSP and FGSP over regular rings: for CSP our results are fairly complete. For FGSP our results cannot be internalized to the ring as well as the domain case. It turns out that torsion submodules of finitely generated modules must be finitely generated, which answers a question about FGSP for regular rings posed by K. Oshiro [12].

The cyclic splitting of special torsion theories, e.g., the Goldie and simple theories, have been extensively studied (see K. Goodearl [8], L. Koifman [11], K. Oshiro [12], J. Pakala [13] and M. Teply [16]). L. Koifman [11] studied CSP in more generality, but still imposed certain primary decomposability constraints on the theories he considered. We synthesize much of the work