Ikeda, T., Sasaki, H. and Yoshida, T.
Osaka J. Math.
22 (1985), 391-392

AN APPLICATION ON NAGAO'S LEMMA

Dedicated to Professor Hirosi Nagao on his sixtieth birthday

TADASI IKEDA, HIROKI SASAKI AND TOMOYUKI YOSHIDA

(Received July 5, 1984)

With time, the importance of Nagao's lemma has grown in modular representation theory of finite groups. In this note, we add another application.

Let G be a finite group, and let F be a field of characteristic $p>0$.

For a subgroup H of G and a (right) FG-module V, we denote V^H the fixed-point-set of H in V, so that V^H is an $FN_G(H)$-module. The trace map $Tr_H^G\colon V^H \to V^G$ is defined by $Tr_H^G(v)=\sum_g vg$, where g runs over a complete set of representatives of $H\backslash G$.

Main Theorem. *Let V be an indecomposable FG-module in a block B, and let P be a p-subgroup of G. Then each composition factor of the $FN_G(P)$-module*

$$V(P) := V^P/\sum_{A<P} Tr_A^P(V^A)\,,$$

where A runs over proper subgroups of P, belongs to a block b such that $b^G=B$.

REMARK. If $V(P)\neq 0$, then P is contained in a defect group of B.

Proof. of the theorem. Set $N=N_G(P)$. Let e be the centrally primitive idempotent of FG corresponding to B. Let $s\colon Z(FG)\to Z(FN)$ be the Brauer homomorphism. Then Nagao's lemma ([2], Chapter III, Theorem 7.5) states that

$$V_N = V_N s(e)\oplus W_1\oplus\cdots\oplus W_n$$

as FN-modules, where each W_i is Q_i-projective FN-module for some p-subgroup Q_i of N with $P\nsubseteq Q_i$. Thus in order to prove the theorem, it will suffice to show that

$$W_i^P \subseteq \sum_{A<P} Tr_A^P(V^A)\,,$$

where A runs over proper subgroups of P. But this follows directly from the following lemma, and so the theorem is proved.

Lemma. *Let N be a finite group with a normal p-subgroup P. Let W be a Q-projective FN-module, where $Q\nsupseteq P$. Then*