Bouleau, N.
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ABOUT STOCHASTIC INTEGRALS WITH RESPECT TO PROCESSES WHICH ARE NOT SEMI-MARTINGALES

NICOLAS BOULEAU

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1. Introduction

Let $(\Omega, \mathcal{F}_t, \boldsymbol{P})$ be a probability space with an increasing right continuous family of $(\mathcal{F}_\infty, \boldsymbol{P})$-complete σ-algebras $(\mathcal{F}_t)$, and let $\mathcal{P}$ be the predictable σ-algebra induced on $\Omega\times\boldsymbol{R}_+$ by the family $(\mathcal{F}_t)$.

For $H\in\mathcal{P}$, we write H_s for the random variable $\omega\rightarrow 1_H(s, \omega)$. If $Z=N+B$ is a semi-martingale such that N is a square integrable martingale and B an adapted process with square integrable variation, the mapping

$$H \rightarrow \int_0^\infty H_s\, dZ_s \tag{1}$$

defines a σ-additive vector measure on $(\Omega\times\boldsymbol{R}_+, \mathcal{P})$ with values in $L^2(\Omega, \mathcal{F}_\infty, \boldsymbol{P})$. It has been shown by several authors that conversely if μ is a σ-additive measure from $\mathcal{P}$ to $L^2(\Omega, \mathcal{F}_\infty, \boldsymbol{P})$ given on the elementary predictable sets H of the form

$$H = h\times]s, t] \qquad 0<s<t\,, \quad h\in\mathcal{F}_s$$

by

$$\mu(H) = 1_h(Z_t-Z_s) \tag{2}$$

for a mean square right-continuous adapted process Z, then there is a modification of Z which is a semi-martingale [2].

Nevertheless, if we consider an other probability space $(W, \mathcal{W}, \boldsymbol{Q})$, an adapted process $(\omega, t)\rightarrow Z_t(\omega, w)$ depending on $w\in W$, and a measure μ which satisfies (2) for elementary predictable sets, and if we replace σ-additivity in $L^2(\boldsymbol{P})$ for each $w\in W$ by σ-additivity in $L^2(\boldsymbol{P}\times\boldsymbol{Q})$, it becomes possible that Z_t fails to be a semi-martingale for fixed w.

In the example that we give, Z_t is, for fixed w, the sum of a martingale and a process of zero energy similar to those considered by Fukushima [3] in order to give a probabilistic interpretation of functions in a Dirichlet space.