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SHARP ESTIMATES FOR THE $\bar{\partial}$-NEUMANN PROBLEM AND THE $\bar{\partial}$-PROBLEM

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0. Introduction

The object of this paper is to establish some estimates for the second order derivatives of the solution of the $\bar{\partial}$-Neumann problem. Similar estimates were obtained by Greiner-Stein [2] when the Levi form is non-degenerate and the metric is a Levi metric. In this article we derive such results merely assuming that the basic estimate (0.2) below holds; the metric may be an arbitrary hermitian metric and we permit some cases where the Levi form is degenerate.

We begin with recalling what the $\bar{\partial}$-Neumann problem is. Let M be a bounded domain in $\boldsymbol{C}^n$ with C^∞-boundary bM. We denote the vector bundle consisting of type (1,0) vectors by S, and the space of smooth (p, q)-forms on $\bar{M}$ by $\mathcal{A}^{p,q}(\bar{M})$. If we write a (p, q)-form ϕ as $\sum'_{I,J} \phi_{I,J}\, dz^I \Lambda \bar{z}^J$, then the $\bar{\partial}$-operator is defined by

$$\bar{\partial}\phi = \textstyle\sum'_{I,J} \sum_{j=1}^{n} \dfrac{\partial \phi_{I,J}}{\partial \bar{z}_j} d\bar{z}_j \Lambda dz^I \Lambda d\bar{z}^J ,$$

where $\{z_1, \cdots, z_n\} = \{x_1+\sqrt{-1}y_1, \cdots, x_n+\sqrt{-1}y_n\}$ is the canonical coordinate system of C^n, $\partial/\partial z_j = \frac{1}{2}(\partial/\partial x_j - \sqrt{-1}\partial/\partial y_j)$, $j=1, \cdots, n$, and the notation $\sum'$ means that the summation is taken over strictly increasing p-tuples I and q-tuples J of $(1, \cdots, n)$. Let $D^{p,q}$ denote the totality of the smooth (p, q)-forms ϕ on $\bar{M}$ such that $(\psi, \vartheta\phi) = (\bar{\partial}\psi, \phi)$ holds for each $\psi \in \mathcal{A}^{p,q-1}(\bar{M})$, where ϑ is the formal adjoint of $\bar{\partial}$ and $(\,,\,)$ the L^2-inner product on M. We consider the following variational problem: given $\lambda \in \boldsymbol{C}$ and $f \in \mathcal{A}^{p,q}(\bar{M})$ arbitrarily, find $u \in D^{p,q}$ such that

$$(0.1)_\lambda \qquad Q(u, \phi) + \lambda(u, \phi) = (f, \phi) \qquad \text{for any } \phi \in D^{p,q} ,$$

where $Q(\phi, \psi) = (\bar{\partial}\phi, \bar{\partial}\psi) + (\vartheta\phi, \vartheta\psi) + (\phi, \psi)$. This problem is equivalent to the following boundary value problem:

$$(0.1)'_\lambda \qquad (\square+\lambda+1)u = f \text{ in } M,\ u \in D^{p,q},\ \bar{\partial}u \in D^{p,q+1} ,$$