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HOMOLOGY LOCALIZATIONS AFTER APPLYING SOME RIGHT ADJOINT FUNCTORS

Dedicated to Professor Nobuo Shimada on his sixtieth birthday

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0. Introduction

Each homology theory E_* determines a natural E_*-localization $\eta: X \to L_E X$ in the homotopy category $h\mathcal{CW}$ of CW-complexes or $h\mathcal{CWS}$ of CW-spectra. It is full of interest to study the behavior of E_*-localizations after application of various functors T to the category $h\mathcal{CW}$ or $h\mathcal{CWS}$. Consider as T the 0-th space functor $\Omega^\infty: h\mathcal{CWS} \to h\mathcal{CW}$ which is right adjoint to the suspension spectrum functor Σ^∞. Bousfield [4] showed that the E_*-localization of an infinite loop space $\Omega^\infty X$ is still an infinite loop space. More precisely, he proved

Theorem 0.1 ([4, Theorem 1.1]). *There exists an idempotent monad L: $h\mathcal{CWS}_0 \to h\mathcal{CWS}_0$ and $\eta: 1 \to L$ such that the map $\Omega^\infty \eta: \Omega^\infty X \to \Omega^\infty LX$ is an E_*-localization in $h\mathcal{CW}$. Here $h\mathcal{CWS}_0$ denotes the full subcategory of $h\mathcal{CWS}$ consisting of (-1)-connected CW-spectra.*

As remarked by Bousfield [4], this implies

Proposition 0.2. *If f: $A \to B$ is an E_*-equivalence in $h\mathcal{CW}$, then so is $\Omega^\infty\Sigma^\infty f$: $\Omega^\infty\Sigma^\infty A \to \Omega^\infty\Sigma^\infty B$.*

On the other hand, Kuhn [7, Proposition 2.4] gave recently a simple proof of Proposition 0.2 using the stable decompositions of $\Omega^\infty\Sigma^\infty A$ and $\Omega^\infty\Sigma^\infty B$ (see [9]).

In this note we will show that Proposition 0.2 is essential to the existence theorem 0.1. Thus, by use of only Proposition 0.2 we give a direct proof of the existence theorem 0.1 along the primary line of Bousfield [1, 2 and 3]. In our proof we don't need the knowledge of very special Γ-spaces although Bousfield did in [4].

Let T: $\mathcal{C} \to \mathcal{B}$ be a functor with a left adjoint S and $\mathcal{W}$ be a morphism class in $\mathcal{B}$. In §1 we introduce $T^*\mathcal{W}$- and $(\mathcal{W}, T)$-localizations in $\mathcal{C}$ and discuss a relation between them. Following our notation Theorem 0.1 says that there exists an (E_*, Ω^∞)-localization in $h\mathcal{CWS}_0$ where E_* stands for the morphism class of E_*-equivalences in $h\mathcal{CW}$. Don't confuse our notation with Bousfield's [4]. We next give three conditions (C.1)–(C.3) under which we can construct