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ACTIONS OF SYMPLECTIC GROUPS ON A PRODUCT OF QUATERNION PROJECTIVE SPACES

Dedicated to Professor Minoru Nakaoka on his 60th birthday

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0. Introduction

We shall study smooth actions of symplectic group $\boldsymbol{Sp}(n)$ on a closed orientable manifold X such that $X \sim \boldsymbol{P}_a(\boldsymbol{H}) \times \boldsymbol{P}_b(\boldsymbol{H})$, under the conditions: $a+b \leqq 2n-2$ and $n \geqq 7$. Our result is stated in §2 and proved in §5. Typical examples are given in §1. Similar result on smooth actions of special unitary group $\boldsymbol{SU}(n)$ on a closed orientable manifold X such that $X \sim \boldsymbol{P}_a(\boldsymbol{C}) \times \boldsymbol{P}_b(\boldsymbol{C})$ is stated in the final section.

Throughout this paper, let $H^*(\ )$ denote the singular cohomology theory with rational coefficients, and let $\boldsymbol{P}_n(\boldsymbol{H})$, $\boldsymbol{P}_n(\boldsymbol{C})$ and $\boldsymbol{P}_n(\boldsymbol{R})$ denote the quaternion, complex and real projective n-space, respectively. By $X \sim X'$, we mean that $H^*(X) \cong H^*(X')$ as graded algebras.

1. Typical examples

1.1. We regard S^{4k-1} as the unit sphere of the quaternion k-space H^k with the right scalar multiplication. Let Y be a compact $\boldsymbol{Sp}(1)$ manifold. By the diagonal action, $\boldsymbol{Sp}(1)$ acts freely on the product manifold $S^{4k-1} \times Y$. Here we consider the cohomology ring of the orbit manifold $(S^{4k-1} \times Y)/\boldsymbol{Sp}(1)$ for the case $Y \sim \boldsymbol{P}_b(\boldsymbol{H})$.

Consider the fibration: $Y \to (S^{4k-1} \times Y)/\boldsymbol{Sp}(1) \to \boldsymbol{P}_{k-1}(\boldsymbol{H})$. By the Leray-Hirsch theorem, $H^*((S^{4k-1} \times Y)/\boldsymbol{Sp}(1))$ is freely generated by $1, u, u^2, \cdots, u^b$ as an $H^*(\boldsymbol{P}_{k-1}(\boldsymbol{H}))$ module for an element $u \in H^4((S^{4k-1} \times Y)/\boldsymbol{Sp}(1))$. If u can be so chosen as $u^{b+1}=0$, then we see that $(S^{4k-1} \times Y)/\boldsymbol{Sp}(1) \sim \boldsymbol{P}_{k-1}(\boldsymbol{H}) \times \boldsymbol{P}_b(\boldsymbol{H})$.

Lemma 1.1. *Denote by F, the fixed point set of the restricted $\boldsymbol{U}(1)$ action on Y. If $F \sim \boldsymbol{P}_b(\boldsymbol{C})$, then $(S^{4k-1} \times Y)/\boldsymbol{Sp}(1) \sim \boldsymbol{P}_{k-1}(\boldsymbol{H}) \times \boldsymbol{P}_b(\boldsymbol{H})$.*

Proof. Consider the fibration: $Y \to (S^{4k-1} \times Y)/\boldsymbol{U}(1) \to \boldsymbol{P}_{2k-1}(\boldsymbol{C})$. We see that $H^*((S^{4k-1} \times Y)/\boldsymbol{U}(1))$ is freely generated by $1, v, v^2, \cdots, v^b$ as an $H^*(\boldsymbol{P}_{2k-1}(\boldsymbol{C}))$ module for an element $v \in H^4((S^{4k-1} \times Y)/\boldsymbol{U}(1))$. We shall show first that