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NONTRIVIALITY OF THE GELFAND-FUCHS CHARACTERISTIC CLASSES FOR FLAT S^1-BUNDLES

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Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

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1. Introduction

Motivated by the work of Gelfand and Fuchs [2], Bott and Haefliger (see [4]) have defined homomorphisms

$$\bar{\Phi}\colon H^*(\mathcal{L}_{S^1}) \to H^*(B\overline{\mathrm{Diff}}_+(S^1); \boldsymbol{R})$$
$$\Phi\colon H^*(\mathcal{L}_{S^1}, so(2)) \to H^*(B\mathrm{Diff}_+(S^1)^\delta; \boldsymbol{R})$$

where $\mathcal{L}_{S^1}$ is the topological Lie algebra consisting of all C^∞ vector fields on S^1, $H^*(\mathcal{L}_{S^1})$ is its continuous cohomology (=the Gelfand-Fuchs cohomology of S^1) and $H^*(\mathcal{L}_{S^1}, so(2))$ is the continuous cohomology of $\mathcal{L}_{S^1}$ relative to the subalgebra $so(2)\subset\mathcal{L}_{S^1}$. $\mathrm{Diff}_+(S^1)$ is the topological group of all orientation preserving C^∞ diffeomorphisms of S^1 and $B\overline{\mathrm{Diff}}^+(S^1)$ (resp. $B\mathrm{Diff}_+(S^1)^\delta$) is the classifying space for the topological group $\overline{\mathrm{Diff}}_+(S^1)$ (=homotopy theoretical fibre of the forgetful homomorphism $\mathrm{Diff}_+(S^1)^\delta\to\mathrm{Diff}_+(S^1)$, here δ denotes the discrete topology) (resp. $\mathrm{Diff}_+(S^1)^\delta$). $B\overline{\mathrm{Diff}}_+(S^1)$ (resp. $B\mathrm{Diff}_+(S^1)^\delta$) classifies foliated S^1-products (resp. foliated S^1-bundles) (see [17]). Gelfand and Fuchs [2] have proved that $H^*(\mathcal{L}_{S^1})$ is a free graded algebra with two generators α of degree 2 and β of degree 3 and it follows that $H^*(\mathcal{L}_{S^1}, so(2))=\boldsymbol{R}[\alpha, \chi]/(\alpha\chi)$ where χ is the Euler class (see [4]). We may call the images of $\bar{\Phi}$ and Φ the Gelfand-Fuchs characteristic classes for flat S^1-bundles. Thurston [16] has constructed examples of foliated S^1-bundles to show that the classes α and χ (we omit the symbols $\bar{\Phi}$ and Φ for simplicity, thus α stands for $\Phi(\alpha)$ for example) are independent and also that all the classes α^n ($n\in\boldsymbol{N}$) vary continuously, namely there are homology classes $\sigma_t\in H_{2n}(B\mathrm{Diff}_+(S^1)^\delta; \boldsymbol{Z})$ with $\langle\sigma_t, \alpha^n\rangle=t$ for all $t\in\boldsymbol{R}$. In this paper we describe an extension of Thurston's argument which proves the nontriviality of the classes $\alpha^{n-1}\beta$ and χ^n ($n\in\boldsymbol{N}$).

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