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THE MINIMUM CROSSING OF 3-BRIDGE LINKS

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1. Introduction

A natural way to simplify a given projection of a link is to decrease the number of crossings of the projection. As a natural expectation, one would like to develop an algorithm to obtain a projection of a link whcih attains the minimum crossing number. Our purpose in this paper is to present such an algorithm for 3-bridge links and to argue on its utility.

A projection $p(L)$ on a plane (or a 2-sphere) of a link L divides into $2n$ arcs of two types; the first types $b_1^+, \cdots, b_n^+$, called *over bridges*, contain only overcrossings of $p(L)$ and the second types $b_1^-, \cdots, b_n^-$, called *under bridges*, contain only undercrossings of $p(L)$. Any two of them are disjoint or intersect transversely in some of the crossings, according to whether they have the same type or not. A link L is called an *n-bridge link* if it admits a projection $p(L)=b_1^+ \cup \cdots \cup b_n^+ \cup b_1^- \cup \cdots \cup b_n^-$ with such a division into $2n$ bridges.

Let ω be an arc on the plane such that for one of bridges of $p(L)$, say b_i^+, $\omega \cap p(L)=\omega \cap b_i^+=\partial\omega$. If the interior of the subarc β of b_i^+ bounded by $\partial\omega$ contains at least one crossing of $p(L)$ then ω is called a *wave* and the replacement of b_i^+ with $(b_i^+-\beta)\cup\omega$ is called a *wave move* with ω. The wave move transforms $p(L)$ into a new projection of L which has fewer crossings than $p(L)$. Note that a wave move decreases the number of crossings but does not change the number of bridges.

In general, a sequence of wave moves does not carry a given projection of a link to a minimum-crossing one. Fig. 1-(a) and (b) show two 3-bridge projections of the trefoil which have no wave. Clearly the number of crossings of the left-hand projection is minimum among *all* of projections of the trefoil. Cancelling a pair of bridges in the right-hand one, we obtain a 2-bridge projection of the trefoil which has no wave. These examples suggest that we can not find in general a *really* minimum-crossing projection of a link, fixing the number of bridges, and in particular by wave moves. Then we shall say that an n-bridge projection is *minimum-crossing*, only meaning that the number of its crossings is minimum among n-bridge projections which the link admits.