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A HOMOTOPY GROUP OF THE SYMMETRIC SPACE $SO(2n)/U(n)$

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In [1] B. Harris calculated some homotopy groups of the symmetric space $\Gamma_n = SO(2n)/U(n)$. He determined $\pi_{2n+r}(\Gamma_n)$ for $-1 \leqq r \leqq 1$ and for $r=3$, $n \equiv 0 \pmod 4$ except for $r=1$, $n \equiv 2 \pmod 4$. For the last case he made a group extension

$$(1) \qquad 0 \to Z_2 \to \pi_{2n+1}(\Gamma_n) \to Z_{n!/2} \to 0$$

from the homotopy exact sequence of the fibration $\Gamma_n \to \Gamma_{n+1} \to S^{2n}$. The purpose of this note is to show that this extension splits.

Theorem. $\pi_{2n+1}(\Gamma_n) = Z_2 \oplus Z_{n!/2}$ *if* $n \equiv 2 \pmod 4$.

Proof. If $n=2$, then the conclusion is obvious, by (1). Thus we will always assume that $n \equiv 2 \pmod 4$ and $n \geqq 6$.

The rotation group $SO(m)$ and the unitary group $U(m)$ are embedded, respectively, in $SO(m+1)$ and $U(m+1)$ as the upper left hand blocks. We embed $U(m)$ in $SO(2m)$ as the subset of matrices consisting of 2×2 blocks

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

The natural map $SO(2n-1)/U(n-1) \to SO(2n)/U(n) = \Gamma_n$ is a homeomorphism and will be used to identify these spaces. The inclusion map $SO(2n-2) \to SO(2n-1)$ then induces a map between the fibrations:

$$\begin{array}{ccc} U(n-1) & = & U(n-1) \\ \downarrow j & & \downarrow \\ SO(2n-2) & \to & SO(2n-1) \\ \downarrow & & \downarrow p \\ \Gamma_{n-1} & \xrightarrow{i} & \Gamma_n . \end{array}$$

Applying the homotopy functor $\pi_*(-)$ to this, we obtain a commutative diagram with exact columns: