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MATRICES OVER GROUP RINGS WHICH ARE ALEXANDER MATRICES[1]

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Introduction

Let $(x_1, \cdots, x_m: r_1, \cdots, r_n)$ be a presentation of a group G. Then an Alexander matrix of G can be obtained by mapping the $n\times m$ matrix $(\partial r_i/\partial x_j)$ into a matrix with coefficients in the group ring JH of some homomorphic image H of G. (We are using i for the row index and j for the column index. Moreover, what we call Alexander matrices are called in Fox [4] 'Homomorphisms of the Jacobian'.) In this note, we consider the reverse of the above procedure. We start with a matrix A over a group ring, and look for groups with an Alexander matrix equal to A.

Let F be the free group on the set of m letters $\{x_1, \cdots, x_m\}$, and JF be the integral group ring on F. Let χ: $F\rightarrow H$ be an epimorphism from F onto a group H, and let $\tilde{\chi}$: $JF\rightarrow JH$ be the extension of χ to group rings. Then for an $n\times m$ matrix A with entries $\tilde{f}^i_j$ over JH, if G is such that

$$\begin{array}{ccc} F & \xrightarrow{\ \chi\ } & H \\ \phi\searrow & & \nearrow\psi \\ & G & \end{array}$$

commutes and $(\partial r_i/\partial x_j)^{\tilde{\chi}}=A$, we say G *realizes* A w.r.t. χ. Here ϕ is the canonical projection, and ψ is the epimorphism induced by χ. Let R denote Ker χ. We show

Theorem I. *Given an $n\times m$ matrix A with entries $\tilde{f}^i_j$ over JH, there is a group G realizing A w.r.t. χ iff $\sum_{j=1}^{m} \tilde{f}^i_j\tilde{\chi}(x_j-1)=0$, $i=1, \cdots, n$. Further if the entries of A satisfy this condition and G is a group with presentation $(x_1, \cdots, x_m: r_1, \cdots, r_n)$ such that $(\partial r_i/\partial x_j)^{\chi}=A$, the collection of all groups realizing A w.r.t χ is*

$$\{(x_1, \cdots, x_m: a_1r_1, \cdots, a_nr_n) \mid a_1, \cdots, a_n\in[R, R]\} .$$

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