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REMARKS ON THE LIFTING PROPERTY OF SIMPLE MODULES

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Throughout this paper, we assume that R is an associative ring with identity and $\{M_\alpha\}_I$ is an infinite set of completely indecomposable right R-modules. We put $M=\sum_I \oplus M_\alpha$ and $\bar{M}=M/J(M)$, where $J(M)$ $(=\sum_I \oplus J(M_\alpha))$ denotes the Jacobson radical of M.

If each M_α is a cyclic hollow module, then $\bar{M}$ is completely reducible. In this case, M is said to have the *lifting property of simple modules modulo the radical* if every simple submodule of $\bar{M}$ is induced from a direct summand of M ([3]). On the other hand, for the family $\mathcal{M}$ of all maximal submodules of M, M is said to have the *lifting property of modules for* $\mathcal{M}$ if every member A in $\mathcal{M}$ is co-essentially lifted to a direct summand of M, that is, there exists a decomposition $M=A^*\oplus A^{**}$ such that $A^*\subseteq A$ and $A\cap A^{**}$ is small in M ([5]). These two concepts are both dual to '*extending property of simple modules*' mentioned in [4]. Therefore, we must observe whether these two lifting properties coincide or not. In this paper, we study this problem and show the following result: M has the lifting property of modules for $\mathcal{M}$ if and only if it has the lifting property of simple modules modulo the radical and satisfies the following condition: For any $\{M_{\alpha_i}\}_{i=1}^{\infty}\subseteq \{M_\alpha\}_I$ and epimorphisms $\{f_i: M_{\alpha_i}\to M_{\alpha_{i+1}}\}_{i=1}^{\infty}$, there exist n (depending on the sets) and epimorphism $g: M_{\alpha_{n+1}}\to M_{\alpha_n}$ such that $\bar{g}=\bar{f}_n^{-1}$, where $\bar{g}$ and $\bar{f}_n$ are the induced isomorphisms: $\bar{M}_{\alpha_{n+1}}\to \bar{M}_{\alpha_n}$ and $\bar{M}_{\alpha_n}\to \bar{M}_{\alpha_{n+1}}$, respectively (Theorem 10).

NOTATION. By $P(M)$ we denote the set of all submodules X of M such that $X\cap M_\alpha \neq M_\alpha$ for all $\alpha\in I$ and $X=\sum_I \oplus(X\cap M_\alpha)$.

We first show

Theorem 1. *The following conditions are equivalent:*

1) *For any pair* $\alpha, \beta\in I$, *every epimorphism from* M_α *to* M_β *is an isomorphism.*

2) *Let* $\{A_\beta\}_J$ *be a family of indecomposable direct summands of* M. *If* $A_{\beta_1}+\cdots+A_{\beta_n}+X\not\supseteq A_{\beta_{n+1}}$ *for any* $X\in P(M)$ *and any finite subset* $\{\beta_1, \cdots, \beta_{n+1}\}$