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DIRECT SUMS OF INDECOMPOSABLE MODULES

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1. Introduction. This paper studies direct sums $M=\oplus_{i\in I}M_i$ of indecomposable modules. Specifically, we give a number of necessary and sufficient conditions for such a sum to be quasi-continuous or continuous. This question was settled in [6], in a very satisfactory way, in case the index set I is finite, or the ring is right-noetherian, but the general case dealt with here is much more complicated.

Such sums $M=\oplus_{i\in I}M_i$ have been investigated in great detail, in a long series of papers since about 1970, by M. Harada and his collaborators, usually under the additional hypothesis that the M_i have local endomorphism rings (so that the Krull-Schmidt-Azumaya Theorem applies). One of the central results is the following:

Theorem 1 ([3], p. 22). *For a module with a decomposition $M=\oplus_{i\in I}M_i$, and with all* endo(M_i) *local, the following statements are equivalent:*

(1) *The decomposition is locally semi-T-nilpotent;*

(2) *it complements direct summands;*

(3) *any local direct summand of M is a direct summand.*

(The relevent terms are defined later on in this section.)

The present paper owes a great deal to the work of these authors. In particular, we refer to [5], [4] and the manuscript [7], results of which are announced in [8]. The reader will notice considerable overlap with the paper by K. Oshiro, and some of our arguments are lifted from it with little modification. Our main original contribution is the application of [1] (cf. our Lemma 1), from which we derive that any quasi-continuous $M=\oplus_{i\in I}M_i$ with indecomposable M_i is locally semi-T-nilpotent, and which allows us to free the results of [7] from their relativization with respect to uniform dimension (ie. the conditions $(\alpha-C_i)$ defined in Section 4), and to simplify the proofs. Moreover, applications of the main theorem of [6] yield a short proof of Theorem 8.

All our modules are right-modules over a ring R. m^0 denotes the annihilator in R, of the element $m\in M$. $X\subset'M$ and $Y\subset^{\oplus}M$ signify that X is an essential submodule, and Y a direct summand, of M. The sum of an independent family of submodules of M is called a local direct summand if every