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$\aleph_0$-CONTINUOUS MODULES

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Generalizing the notion of right $\aleph_0$-continuous regular rings (see [2], [3]) we define that of quasi-$\aleph_0$- and $\aleph_0$-continuous modules and mainly study the directly finiteness of nonsingular $\aleph_0$-continuous modules over (von Neumann) regular rings.

Let R be a regular ring. By $\mathcal{F}$ we denote the family of all essentially $\aleph_0$-generated essential right ideals of R. It is shown that $\mathcal{F}$ becomes a right Gabriel topology on R (Proposition 5). From this fact the divisible hull $E_{\mathcal{F}}(M)$ of a given right R-module M is considered. Our main purpose of this note is to prove that a nonsingular $\aleph_0$-continuous R-module M is directly finite if and only if so is $E_{\mathcal{F}}(M)$. This is a generalization of a result due to Goodearl [3].

Throughout this paper R is a ring with identity and all R-modules considered are unitary right R-modules.

For a given R-module M, we denote its injective hull by $E(M)$ and the family of all submodules of M by $\mathcal{L}(M)$.

For $N\in\mathcal{L}(M)$ $N\leqq_e M$ means that N is an essential submodule of M and $(N: x)$, for $x\in M$, denotes the right ideal $\{r\in R\,|\,xr\in N\}$.

Let M be an R-module. An $\mathcal{S}$-closed submodule of M is a submodule B such that M/B is nonsingular. For any submodule A of M there exists the smallest $\mathcal{S}$-closed submodule C of M containing A, which is called the $\mathcal{S}$-closure of A in M (see [1]). We note that, when M is nonsingular, the $\mathcal{S}$-closure C of A in M is uniquely determined as a submodule C such that $A\leqq_e C$ and C is $\mathcal{S}$-closed in M.

Lemma 1. *Let M be an R-module, and let A and B be submodules of M such that $A\leqq_e B$. Then B is contained in the $\mathcal{S}$-closure of A in M. In addition if M is nonsingular and B is a direct summand of M, then B coincides with the $\mathcal{S}$-closure of A in M.*

Proof. Let C be the $\mathcal{S}$-closure of A in M. Since $(B+C)/C$ is an epimorphic image of a singular module B/A, we see that $(B+C)/C$ is singular. On the other hand, $(B+C)/C$ is a submodule of a nonsingular module M/C,