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ORTHOGONAL GROUPS AND SYMMETRIC SETS

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Orthogonal groups are considered as automorphism groups of some symmetric sets of vectors. From this point of view, we can prove the well-known theorem of simplicity on orthogonal groups. (The cases for the other classical groups are given in [5].) The proof consists of two steps. The first step which will be given in **1** is to show that a transitive symmetric set of non-isotropic lines (of a certain type) is simple. After a short review on simple symmetric set is given, we will show the above fact. A point here is that it is so when dim V is 3. The second step is to show that the group of displacements of the simple symmetric set is a simple group, which will be given in **2**. A useful supplement to the main theorem on simple symmetric sets will be found, and using it we can show the above fact when dim $V \geq 5$.

1. A simple symmetric set of non-isotropic lines

Let V be a vector space over a field of characteristic $\neq 2$ with a non-singular orthogonal metric. Since the following results hold in a stronger sense for a finite field as was shown in [4], we assume in this note that the base field k is infinite. Suppose that dim $V \geq 3$ and that V contains a hyperbolic plane. Then, there exists a vector v such that v is orthogonal to a hyperbolic plane and $(v, v)=\varepsilon \neq 0$. Throughout this note, we fix the element ε. Now we consider $A=\{\bar{u} \mid (u, u)=\varepsilon\}$, where $\bar{u}=\langle u\rangle=$a subspace generated by u. On A, we define a binary operation: $\bar{u}\circ\bar{v}=\bar{w}$ with $w=u^{\tau_v}$, where τ_v is the symmetry with respect to the hyperplane orthogonal to v. A is then a symmetric set, i.e., satisfies $\bar{u}\circ\bar{u}=\bar{u}$, $(\bar{u}\circ\bar{v})\circ\bar{v}=\bar{u}$ and $(\bar{u}\circ\bar{v})\circ\bar{w}=(\bar{u}\circ\bar{w})\circ(\bar{v}\circ\bar{w})$.

We summarize some definitions and properties on simple symmetric sets. Let $S=\{a, b, c, \cdots\}$ be a symmetric set. The right multiplication by an element a is an automorphism of S, which we denote by σ_a. Let $G(S)=\langle\sigma_a \mid a\in S\rangle$ and $H(S)=\langle\sigma_a\sigma_b \mid a, b\in S\rangle$. The latter is called the group of displacements of S. Let T be another symmetric set. A homomorphism f of S onto T is called proper if it is not one to one and if T contains more than one element. When $G(S)=1$, we say S is trivial. A non-trivial symmetric set is called simple if it has no proper homomorphism (to some symmetric set). It is important