ON M-RINGS AND GENERAL ZPI-RINGS

Dedicated to Professor Kentaro Murata on his 60th birthday

TAKASABURO UKEGAWA

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In the preceding paper [10], we have proved that a left Noetherian $M$-ring is a so called "general ZPI-ring" in the commutative case. Also we know that in an $M$-ring the multiplication of prime ideals is commutative [8]. In the present paper we define general ZPI-rings in section 1 and we study general properties of them, and as an important example of such rings we can give a left Noetherian semi-prime Asano left order. In section 2 we research the condition for a left Noetherian general ZPI-ring to be an $M$-ring, using minimal prime divisors of an ideal. The notation "<" means a proper inclusion as the preceding papers [8], [9], [10].

1. $M$-rings and general ZPI-rings

**Definition.** If the multiplication of any two prime ideals of a ring $R$ is commutative, and any ideal of $R$ can be written as a product of powers of prime (considering $R$ as a prime ideal) ideals of $R$, then we call $R$ a general ZPI-ring. Therefore the multiplication of ideals is commutative.

In the commutative case a general ZPI-ring is necessarily Noetherian no matter whether the ring has an identity or not. But in our case the general ZPI-ring is not necessarily Noetherian as the example in [9] shows.

**Proposition 1.** Let $R$ be a left Noetherian general ZPI-ring, let $P$ be any prime ideal of $R$, and let $q$ be maximal in the set of prime ideals such that $q<P$. Then for any ideal $a$ with $q<a<P$, there is an ideal $b$ such that $a=Pb=bP$.

**Proof.** Let $a=\prod p_i < P$, since $R$ is a general ZPI-ring. Then $p_i \subseteq P$ for some $p_i$. Since $q<a \leq p_i$, $q<p_i \subseteq P$, so $p_i = P$. Therefore $a=\prod p_i \cdots p_{i+1} \cdots p_s = \prod p_i \cdots p_{i-1} p_{i+1} \cdots p_s$, where $b=\prod p_i \cdots p_{i-1} p_s$.

As in the commutative case we have

**Proposition 2.** Let $R$ be a left Noetherian general ZPI-ring, and let $P$ be a maximal ideal of $R$. Then there are no ideals between $P$ and $P^2$ (including the case that $P=P^2$), more generally for any positive integer $n$, the only ideals