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STOCHASTIC INTERSECTION NUMBER AND HOMOLOGICAL BEHAVIORS OF DIFFUSION PROCESSES ON RIEMANNIAN MANIFOLDS

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1. Introduction. Let M be a d-dimensional ($d \geqq 2$) connected orientable Riemannian manifold and g be its Riemannian metric. Let Δ be the Laplace-Beltrami operator on M and b be a C^∞ vector field on M. Set $L=\Delta/2+b$. Let $X=(X_t, P_x, x \in M)$ be the minimal diffusion process corresponding to L. The main purpose of the present paper is to investigate some homological behaviors of the path of X. Our first objective is to define the intersection number of the path of X and chains and to study its asymptotic behaviors. The usual intersection number of two cycles is defined as the product of their homology classes. But we need in our study the intersection number for chains and its analytical expressions. Theory of harmonic integrals gives such an expression. In passing from smooth cases to stochastic cases, we use the so-called Fisk-Stratonovich integral, which enables us to write down the formulas formally in the same way as in ordinary calculus. Based on the analytic expression of the intersection numbers expressed by the integral of double 1-form with singularity, we shall define the stochastic intersection number with the aid of the integral of 1-form along the path ([8]). By virtue of the approximation theorem for stochastic integrals ([8]), it turns out that stochastic intersection number enjoys analogous properties to those for ordinary intersection numbers. In the study of the asymptotic behavior of the intersection numbers of the path and the cycles, the integrals of harmonic 1-forms play an important role, which is due to the two facts that they depend only on the homology class of the path and that they are martingales.

We then consider the following problem related to the asymptotic behaviors of the intersection numbers. Let M be two-dimensional, compact and let κ be its genus ($1 \leqq \kappa < \infty$). Let X be a Brownian motion on M. Our problem is this. In what manner does the path of X wind holes asymptotically? We formulate this as follows. Let $(A_i, A_{\kappa+i})$, $i=1, \cdots, \kappa$, be a canonical homology basis. For any $x, y \in M$, we choose a smooth curve $\phi_{x,y}$ such that $\phi_{x,y}(0)=x$, $\phi_{x,y}(1)=y$ and $\phi_{x,y}(0,1)$ does not meet any $A_i (i=1, \cdots, 2\kappa)$. Set $C=\{\phi_{x,y};\, x, y \in M\}$. Consider the homological position of the curve $X[0,t]$, that is, using $\phi_{X(t),X(0)}$, we

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