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THE GROUP OF UNITS OF THE INTEGRAL GROUP RING OF A METACYCLIC GROUP

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We denote by $U(\Lambda)$ the group of units of a ring Λ. Let G be a finite group and let ZG be its integral group ring. Define $V(ZG)=\{u\in U(ZG)\,|\,\varepsilon(u)=1\}$ where ε denotes the augmentation map of ZG. In this paper we will study the following

Problem. *Is there a torsion-free normal subgroup F of $V(ZG)$ such that $V(ZG)=F\cdot G$?*

Denote by S_n the symmetric group on n symbols, by D_n the dihedral group of order $2n$ and by C_n the cyclic group of order n. The problem has been solved affirmatively in each of the following cases:

(1) G an abelian group (Higman [4]),

(2) $G=S_3$ (Dennis [2]),

(3) $G=D_n$, n odd (Miyata [5]) or

(4) G a metabelian group such that the exponent of G/G' is 1, 2, 3, 4 or 6 where G' is the commutator subgroup of G ([7]).

The purpose of this paper is to solve the problem for a class of metacyclic groups. Our main result is the following

Theorem. *Let $G=C_n\cdot C_q$ be the semidirect product of C_n by C_q such that $(n,q)=1$, q odd, and C_q acts faithfully on each Sylow subgroup of C_n. Then there exists a torsion-free normal subgroup F of $V(ZG)$ such that $V(ZG)=F\cdot G$.*

1. Lemmas

We begin with

Lemma 1.1. *Let r, k, n be non negative integers and h be a positive integer. Then*

(1) $\sum_{r=0}^{n}(r+1)\cdots(r+k)=(n+1)\cdots(n+k+1)/(k+1)$, *and*

(2) $$\sum_{r=0}^{n}r^h(r+1)\cdots(r+k)=\frac{n(n+1)\cdots(n+k+1)f(n,k,h)}{(k+2)\cdots(k+h+1)},$$