Nobusawa, N.
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A REMARK ON CONJUGACY CLASSES IN SIMPLE GROUPS

NOBUO NOBUSAWA

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Let A be a union of some conjugacy classes in a group. We define a binary operation on A by $a\circ b=b^{-1}ab$. It satisfies that (1) $a\circ a=a$, (2) $(a\circ b)\circ c=(a\circ c)\circ(b\circ c)$ and (3) a mapping σ_a: $x\rightarrow x\circ a$ is a permutation on A. Generally we call a binary system which satisfies the above three conditions a pseudosymmetric set. It is called a symmetric set if (4) σ_a has the order 2 is also satisfied. The set of all nilpotent elements in a Lie algebra is another example of a pseudosymmetric set, where $\sigma_a=\exp(\mathrm{ad}\, a)$. The purpose of this note is to generalize the main result on the simplicity of a symmetric set given in [2] to the case of a pseudosymmetric set. As applications, three examples of conjugacy classes in simple groups A_n, $SL(V)$ and $Sp(V)$ will be discussed, from which we could derive a new proof of the simplicity of the corresponding groups A_n, $PSL(V)$ and $PSp(V)$.

Generally, let A be a pseudosymmetric set and define $G=G(A)=\langle\sigma_a|a\in A\rangle$, a group generated by σ_a. The above three conditions imply that G is a group of automorphisms of A. Note that if ρ is an automorphism of A, then $\sigma_{a^\rho}=\rho^{-1}\sigma_a\rho$. $\{\sigma_a|a\in A\}$ is a union of conjugacy classes in G and hence is a pseudosymmetric set, and the mapping σ: $a\rightarrow\sigma_a$ is a homomorphism of A to the set. When σ is a monomorphism, we say that A is effective. When $A=a^G$ for an element a, we say that A is transitive. Let G' be the commutator subgroup of G. When A is transitive, $G'=\langle\sigma_a^{-1}\sigma_b|a, b\in A\rangle$, since $b=a^\rho$ with some element ρ in G and $\sigma_a^{-1}\sigma_b=\sigma_a^{-1}\rho^{-1}\sigma_a\rho\in G'$ and conversely $\sigma_a^{-1}\sigma_b^{-1}\sigma_a\sigma_b=\sigma_a^{-1}\sigma_c$ with $c=a^{\sigma_b}$. So, in this case, $G=\langle G', \sigma_a\rangle$ for any a. Also note that if A is a union of conjugacy classes in a group K and if A generates K, then $G\cong K/Z(K)$, where $Z(K)$ is the center of K.

Let A and B be pseudosymmetric sets and suppose that there exists a homomorphism f of A onto B. The inverse image $f^{-1}(b)$ for an element b in B is called a coset of f. Let $\{C_i\}$ be the set of all cosets of f. Then $\{C_i\}$ is a system of blocks of imprimitivity of the permutation group G, and if σ and ρ belong to the same coset, then $C_i^\rho=C_i^\sigma$ for every i. When $|B|>1$ and f is not a monomorphism, we say that f is proper. A pseudosymmetric set A with $|A|>2$ is called simple if it has no proper homomorphism. Note that if A is simple, then it is transitive. For, consider the canonical homomorphism $a\rightarrow a^G$