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THE COHOMOLOGICAL ASPECTS OF HOPF GALOIS EXTENSIONS OVER A COMMUTATIVE RING

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Introduction. Let R be a commutative ring with identity, H a finite co-commutative Hopf algebra over R and A an H-Hopf Galois extension of R in the sense of [15]. When R is a field and H is a group ring RG, H-module structure is simply stated as "*the normal basis theorem*" and combined with the theory of Galois algebras [8], [9]. But the normal basis theorem heavily depends on the RG-isomorphism $\mathrm{Hom}_R(RG, R) \cong RG$. Therefore, in considering Hopf Galois extensions, the corresponding notion would be *the dual normal basis theorem*- an H-Hopf Galois extension of R is isomorphic to $H^* = \mathrm{Hom}_R(H, R)$ as H-modules - of course this does not always hold. We shall call such one a Hopf Galois extension with a dual normal basis. On the other hand, A. Nakajima [12], [13] examined an H-module structure under rather strong assumption $H^* \cong H$ and obtained information concerning the relation between the generalized Harrison cohomology groups and Hopf Galois extensions.

In this paper, we shall examine an H-module structure of Hopf Galois extensions and then shall establish a exact sequence involving the isomorphism classes of Hopf Galois extensions, unit-valued Harrison cohomology groups and Pic-valued Harrison cohomology groups, but unfortunately we must essentially assume that H is commutative for a cohomological nature. In §1, we shall prove that an H-Hopf Galois extension A has a decomposition $A \cong H^* \otimes_H P$ as left H-modules with a rank 1 H-projective module P satisfying some cohomological properties. In §2, we deal with Hopf Galois extensions with a dual normal basis. In §3, we shall start from a rank 1 H-projective module P with further cohomological properties and then construct a Hopf Galois extension of R from P. Finally in §4, using the results of §1, §2 and §3, we shall show that the isomorphism classes of Hopf Galois extensions of R forms an abelian group. In Appendix, we shall define the generalized Harrison cohomology groups (c.f. [12]) and then, following the idea of A. Hattori [6], [7] we construct the cohomology groups $\boldsymbol{H}^n(H)$ related to the generalized Harrison cohomology groups. Also we show that $\boldsymbol{H}^2(H)$ is isomorphic to the group of isomorphism classes of H-Hopf Galois extensions of R using the results of