Weinstock, B. M.
Osaka J. Math.
17 (1980), 763-767

ON A THEOREM OF A. SAKAI

BARNET M. WEINSTOCK[1)]

(Received May 7, 1979)

Let K be a compact subset of $\boldsymbol{C}^n$. Let $R_1, \cdots, R_s$, $0<s<n$, be continuous, complex-valued functions on K, each of which can be extended to a neighborhood U of K so as to be holomorphic in $z_{s+1}, \cdots, z_n$. Let A denote the algebra of continuous functions on K which can be approximated uniformly on K by polynomials in $z_1, \cdots, z_n, \bar{z}_1+R_1, \cdots, \bar{z}_s+R_s$. Clearly A is a subalgebra of the algebra B of continuous functions on K which can be approximated uniformly on K by functions which can be extended to some neighborhood of K so as to be holomorphic in $z_{s+1}, \cdots, z_n$. The goal of this paper is the following theorem which gives sufficient conditions for the equality of these two algebras.

Theorem. *Assume that $R_1, \cdots, R_s$ satisfy*

$$(*) \qquad \sum_{j=1}^{s} |R_j(z+w)-R_j(z)|^2 \leq k \sum_{j=1}^{s} |w_j|^2$$

for all $a \in U$ and all w such that $z+w \in U$ and $w_{s+1}=\cdots=w_n=0$, with $0 \leq k<1$. Assume further that for each $z' \in \boldsymbol{C}^s$ the set $K_{z'}=\{z'' \in \boldsymbol{C}^{n-s}\colon (z', z'') \in K\}$ is polynomially convex. Then $A=B$.

This theorem was formulated and proved by A. Sakai [4] undei the further assumption that $R_1, \cdots, R_s \in C^\infty(U)$. The special case when $s=n-1$ and $R_1, \cdots, R_{n-1}$ vanish identically was established much earlier by W. Rudin [3].

Sakai's proof is based on the method used by L. Hörmander and J. Wermer [2] who considered the case $s=n$ (where, of course, the sets $K_{z'}$ play no role) under the assumption that the functions $R_1, \cdots, R_n$ are differentiable of sufficiently high order. Our proof depends instead on the Cauchy-Fantappiè integral techniques used by the author [5, 6] to prove the Hörmander-Wermer theorem with minimal smoothness hypotheses, and also on Rudin's argument for the special case cited in the previous paragraph. More specifically, we use an argument due to Rudin to reduce the proof of the theorem to the assertion that if $h \in C_0^1(\boldsymbol{C}^s)$, and $\tilde{h}(z', z'')=h(z')$ then $\tilde{h}|K \in A$. The assertion is then proved using the Cauchy-Fantappiè formula as in the Appendix of [6].

---

1) This research was partialy supported by a grant from the National Science Foundation.