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ON THE LEAST POSITIVE EIGENVALUE OF LAPLACIAN FOR COMPACT HOMOGENEOUS SPACES

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Introduction and statement of results

Let M be an n-dimensional compact smooth manifold. Two Riemannian metrics g_1 and g_2 on M are called to be *homothetically equivalent* if there exists a diffeomorphism Φ of M onto itself such that $\Phi^* g_1$ coincides g_2 with a constant multiple.

Let $M=G/K$ be a compact homogeneous space, where G is a compact Lie group and K is a closed subgroup of G. A Riemannian metric g on M is called be *G-invariant* if all the translations τ_x by elements x in G on M are isometric with respect to the metric g (cf. [3]). Let us consider the elementary, but non-trivial problem: *How many G-invariant mutually homothetically inequivalent Riemannian metrics are there on $M=G/K$?*

If the linear isotropy action of K on the tangent space $T_o(M)$ of M at the origin $o=\{K\}\in M$ (cf. [3]) is irreducible over $\boldsymbol{R}$, then there exists a unique (up to homothetic equivalence) G-invariant Riemannian metric on M (cf. [9]). So the above problem is reduced to the case that the linear isotropy action of K is *reducible* over $\boldsymbol{R}$, that is, the tangent space $T_o(M)$ is decomposed into two proper subspaces invariant by the linear isotropy action of K. In this case, many people would have the following conjecture: *If a compact homogeneous space $M=G/K$ (with some additional assumptions) has the reducible isotropy action of K over $\boldsymbol{R}$, then it would have uncountably many mutually homothetically inequivalent G-invariant metrics.*

One of the purposes of this paper is to show that the above conjecture is affirmative.

Now we assume that a compact homogeneous space G/K has the condition (C): The linear isotropy action of K on the tangent apace $T_o(M)$ of M at the origin o is reducible and includes the identity representation of K on $T_o(M)$. Let $\mathfrak{g}$ be the Lie algebra of all left invariant vector fields on G and let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ corresponding to the subgroup K. Since G is compact, there exists an $\mathrm{Ad}(G)$-invariant inner product B on $\mathfrak{g}$. Let $\mathfrak{m}$ be the ortho-complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. Then we have the decomposition