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LEFT NOETHERIAN MULTIPLICATION RINGS

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Recently in [9], we defined a multiplication ring, shortly an M-ring, as a ring such that for any two ideals $\mathfrak{a}$, $\mathfrak{b}$ with $\mathfrak{a}<\mathfrak{b}$, there exists $\mathfrak{c}$, $\mathfrak{c}'$ such that $\mathfrak{a}=\mathfrak{b}\mathfrak{c}=\mathfrak{c}'\mathfrak{b}$, where "$<$" means a proper inclusion. An M-ring R is called to be *non-idempotent* if $R>R^2$, and now we call an M-ring R with $R=R^2$ *idempotent*. In [9] we have proved that the unique maximal idempotent ideal $\mathfrak{d}$ of a non-idempotent M-ring can be obtained as an intersection of some sequence of ideals $\{\mathfrak{d}_\alpha\}_\Lambda$ ([9], Theorem 5): $\mathfrak{d}=\bigcap_{\alpha\in\Lambda}\mathfrak{d}_\alpha$. In this note we shall prove that any left Noetherian M-ring is a so-called "general ZPI-ring" in a commutative case ([2]), i.e. each ideal can be written as a product of prime ideals, and as a consequence the multiplication of ideals is commutative. §1 is preliminaries of our study, and in §2 we consider left Noetherian non-idempotent M-rings, and in §3 we study an idempotent case, and the last chapter is the summary of our study. In the following arguments we do not assume the existence of the identity.

1. Preliminaries

In general, we have the following:

Theorem 1. *Let R be an M-ring, and let $\mathfrak{a}$, $\mathfrak{b}$ be two idelas of R with $\mathfrak{a}<\mathfrak{b}$. Then there exists the unique maximal element in the set of ideals $\mathfrak{c}$ with $\mathfrak{a}=\mathfrak{b}\mathfrak{c}$ and also in the set of ideals $\mathfrak{c}'$ of R with $\mathfrak{a}=\mathfrak{c}'\mathfrak{b}$.*

Proof. Let M denote the set $\{\mathfrak{c}$: ideals of R with $\mathfrak{a}=\mathfrak{b}\mathfrak{c}\}$. By Zorn's Lemma, there exist maximal elements in M. Let $\mathfrak{c}_1$, $\mathfrak{c}_2$ be two maximal elements in M. In this case $\mathfrak{a}=\mathfrak{b}\mathfrak{c}_1$ and $\mathfrak{a}=\mathfrak{b}\mathfrak{c}_2$. So $\mathfrak{a}=(\mathfrak{a}, \mathfrak{a})=(\mathfrak{b}\mathfrak{c}_1, \mathfrak{b}\mathfrak{c}_2)=\mathfrak{b}(\mathfrak{c}_1, \mathfrak{c}_2)$. Obviously $(\mathfrak{c}_1, \mathfrak{c}_2)\supseteq\mathfrak{c}_1, \mathfrak{c}_2$. Since both $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are maximal in M, we have $\mathfrak{c}_1=(\mathfrak{c}_1, \mathfrak{c}_2)=\mathfrak{c}_2$.

DEFINITION. We shall call a prime ideal $\mathfrak{p}$ of R a minimal prime divisor of an ideal $\mathfrak{a}$ if $\mathfrak{a}\subseteq\mathfrak{p}$ and there is no prime ideal $\mathfrak{p}'$ with $\mathfrak{a}\subseteq\mathfrak{p}'<\mathfrak{p}$. (c.f. [6]).

Lemma 1. *Let R be an M-ring and $\mathfrak{a}$, $\mathfrak{b}$ two ideals of R with $\mathfrak{a}<\mathfrak{b}$. If $\mathfrak{b}$ is an idempotent ideal, then $\mathfrak{a}=\mathfrak{a}\mathfrak{b}=\mathfrak{b}\mathfrak{a}$.*