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AN EXAMPLE OF A RING WHOSE PROJECTIVE MODULES HAVE THE EXCHANGE PROPERTY

Dedicated to Professor Goro Azumaya on his 60th birthday

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Harada and Ishii [2] and Yamagata [5] have shown the following result: If R is a right perfect ring, then every projective right R-module has the exchange property, and in the case when R is a direct sum of indecomposable right ideals, the converse also holds. In this paper, we show that this converse statement does not hold in general by showing the following theorem: If R is a Boolean ring whose socle is a maximal ideal, then every projective R-module has the exchange property. It should be noted that Nicholson [3] has also shown that every projective right module over a right perfect ring has the finite exchange property, and he then asked whether the converse hold or not. Our theorem answers this question in the negative.

Throughout this paper we assume that a ring R has identity 1 and all R-modules are unitary.

A right R-module M has the *exchange property* if for any right R-module X and any two decompositions

$$X = M'\oplus N = \sum_{\lambda\in\Lambda} \oplus A_\lambda$$

where $M'\approx M$, there exist submodules $A'_\lambda\subseteq A_\lambda$ such that

$$X = M'\oplus(\sum_{\lambda\in\Lambda}\oplus A'_\lambda)\,.$$

M has the *finite exchange property* if this holds whenever the index set Λ is finite.

For a given projective right R-module P, the following condition (N) due to Nicholson [3] seems to be useful for the study of the exchange property:

(N) If $P=\sum_{\lambda\in\Lambda} P_\lambda$, where P_λ are submodules, there exists a decomposition $P=\sum_{\lambda\in\Lambda}\oplus P'_\lambda$ with $P'_\lambda\subseteq P_\lambda$ for each $\lambda\in\Lambda$.

Lemma 1. *If R is a right hereditary ring whose projective right modules satisfy the condition* (N), *then every projective R-module has the exchange property.*