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PRIMITIVE SYMMETRIC SETS IN FINITE ORTHOGONAL GEOMETRY

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Let V be a vector space over a finite field k of characteristic $\neq 2$, and (x, y) a non-degenerate symmetric bilinear form on V. For an element a in V with $(a, a)\neq 0$, we denote by σ_a the reflection in the hyperplane orthogonal to a. A subspace generated by $a, b, \cdots, c$ is denoted by $\langle a, b, \cdots, c\rangle$. Especially $\langle a\rangle$ is denoted by $\bar{a}$. Let $A=\{\bar{a}\mid(a, a)=1\}$. We can define a symmetric structure on A by $\bar{a}\circ\bar{b}=\bar{c}$, where $c=a^{\sigma_b}$. The main object of this note is to show that if dim $V>4$ or if dim $V=4$ and $k\neq F_3$ (the field of three elements), then A is a primitive symmetric set. For the primitive symmetric set, see [3]. Group-theoretically this implies that the centralizer of the involution σ_a in the orthogonal group is a maximal subgroup.

Let $G(V)$ be the orthogonal group, and Ω its commutator subgroup. Let $H(A)$ be the group generated by $\sigma_a\sigma_b$ where $(a, a)=(b, b)=1$. Note that the restriction of $H(A)$ onto A is called the group of displacements and is denoted by $H(A)$ in the previous papeıs. We denote the latter by $\bar{H}(A)$.

Lemma 1. *Suppose that dim $V\geq 4$. Let a and b be elements in V such that $(a, a)=(b, b)\neq 0$ and that $\langle a, b\rangle$ is a non-singular subspace of dim 2. If x is an element in V such that $(x, x)=(a, a)$ and dim $\langle a, x\rangle=2$, then there exist τ_1 and τ_2 in $G(V)$ and c in V such that $a^{\tau_1}=a$, $x^{\tau_1}=c$, $a^{\tau_2}=b$ and $x^{\tau_2}=c$.*

Proof. First, we note that if y and z are elements in V such that $(y, y)=(z, z)\neq 0$ and that dim $\langle y, z\rangle=2$, then $\langle y, z\rangle$ is non-singular if and only if $(y, z)\neq\pm(y, y)$. For, let $z=\alpha y+t$ with α in k and t in V such that $(y, t)=0$ and $t\neq 0$. Then $\langle y, z\rangle$ is singular if and only if $(t, t)=0$, if and only if $\alpha=\pm 1$, if and only if $(y, z)=\pm(y, y)$. Now, put $c=\beta(a+b)+u$ with β in k and u in V such that $u\in\langle a, b\rangle^{\perp}$. We let $\beta=(a, x)((a, a)+(a, b))^{-1}$. This is possible since $(a, a)\neq-(a, b)$ as noted first. Then $(a, c)=(b, c)=(a, x)$. Next, select u suitably in $\langle a, b\rangle^{\perp}$ so that $(c, c)=(a, a)$. Thıs is possible since $\langle a, b\rangle^{\perp}$ is universal, *i.e.*, $k=\{(u, u)\mid u\in\langle a, b\rangle^{\perp}\}$. Note dim $V\geq 4$ and hence dim $\langle a, b\rangle^{\perp}\geq 2$. Thus, we have $\langle a, x\rangle\cong\langle a, c\rangle\cong\langle b, c\rangle$, the first elements corresponding to the first, and the second to the second by the isomorphisms. Then by Witt's theorem, we have the consequence stated in Lemma 1.