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GENERICALLY RATIONAL POLYNOMIALS

Dedicated to Professor Y. Nakai on his sixtieth birthday

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Introduction. Let k be an algebraically closed field of characteristic zero. Let $k[x,y]$ be a polynomial ring of two variables and let $\boldsymbol{A}_k^2=\mathrm{Spec}(k[x,y])$. Embed $\boldsymbol{A}_k^2$ into the projective plane $\boldsymbol{P}_k^2$ as the complement of a line l_∞. Let $f\in k[x,y]$ be an irreducible polynomial, let F_α be the curve on $\boldsymbol{A}_k^2$ defined by $f=\alpha$ for every $\alpha\in k$ and let C_α be the closure of F_α in $\boldsymbol{P}_k^2$. Then the set $\Lambda(f):=\{C_\alpha;\ \alpha\in k\cup(\infty)\}$ is a linear pencil on $\boldsymbol{P}_k^2$ defined by f, where $C_\infty=dl_\infty$, d being the degree of f. The set $\Lambda_0(f):=\{F_\alpha;\ \alpha\in k\}$ is called *the linear pencil on $\boldsymbol{A}_k^2$ defined by f*. The polynomial f is called *generically rational* when the general members of $\Lambda(f)$ (or $\Lambda_0(f)$) are irreducible rational curves. Since the algebraic function field $k(x,y)$ of one variable over the subfield $k(f)$ then has genus 0, Tsen's theorem says that f is generically rational if and only if f is a field generator in the sense of Russell [9, 10], i.e., there is an element $g\in k(x,y)$ such that $k(x,y)=k(f,g)$. If f is a generically rational polynomial, we can associate with f a non-negative integer n, where $n+1$ is the number of places at infinity of a general member F_α of $\Lambda_0(f)$, i.e., the number of places of F_α whose centers lie outside $\boldsymbol{A}_k^2$.

If $d\geqq 1$, the pencil $\Lambda(f)$ has base points situated outside $\boldsymbol{A}_k^2$. Let $\varphi\colon W\to\boldsymbol{P}_k^2$ be the shortest succession of quadratic transformations with centers at the base points (including infinitely near base points) of $\Lambda(f)$ such that the proper transform Λ' of $\Lambda(f)$ by φ has no base points. Then the linear pencil Λ' defines a morphism $\rho\colon W\to\boldsymbol{P}_k^1$, whose general fibers are the proper transforms of general members of $\Lambda(f)$; thence they are nonsingular rational curves by virtue of Bertini's theorem. Moreover, W contains in a canonical way an open subset isomorphic to $\boldsymbol{A}_k^2$. A generically rational polynomial f is said to be *of simple type* if the morphism ρ has $n+1$ cross-sections contained in the boundary set $W-\boldsymbol{A}_k^2$ (cf. Definition 1.8, below).

If $n=0$, a generically rational polynomial f is sent to one of the coordinates x, y of $\boldsymbol{A}_k^2$ by a biregular automorphism of $\boldsymbol{A}_k^2$ (cf. Abhyankar-Moh's theorem [1, 4]); hence f is of simple type. If $n=1$, a generically rational polynomial is always of simple type (cf. Theorem 2.3, below). However, if $n>1$, a generi-

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