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CONFORMAL STRUCTURES ON THE REAL n-TORUS

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0. Introduction

As is well known, a doubly connected region B on the complex projective line $\boldsymbol{P}^1(\boldsymbol{C})$ can be mapped conformally onto the interior of a concentric annular region lying on $\boldsymbol{P}^1(\boldsymbol{C})$ with radii r_1, r_2 $(0 \leq r_1 < r_2 \leq \infty)$. The quantity $\log (r_2/r_1)$, completely determined by B, is called *conformal modulus* of B. A pair of like regions B, B' is conformally equivalent if and only if their moduli are equal. Furthermore the conformal modulus is continuous to the effect that a proximity of B' to B in a certain sense (for example, with respect to Fréchet metric) implies the smallness of difference between their moduli.

Suppose given a real 2-torus $\boldsymbol{T}^2 = S^1 \times S^1$. The universal covering surface $\tilde{T}^2$ of $\boldsymbol{T}^2$, which is conformally equivalent to $\boldsymbol{C}$ gives rise to the group of its cover transformations $z \mapsto z + m_1\omega_1 + m_2\omega_2$ with a pair of complex constants ω_1, ω_2 called canonical periods and with $m_1, m_2 = 0, \pm 1, \pm 2, \cdots$. One adopts the ratio $\tau = \omega_2/\omega_1$ as a *conformal modulus* for $\boldsymbol{T}^2$. But, because all the modular transformation $J(\tau)$ of τ corresponds to the same $\boldsymbol{T}^2$, τ is not uniquely determined by $\boldsymbol{T}^2$. That will be the naivest aspect of what we understand and should expect under the terminology *conformal modulus* (cf. e.g., Oikawa [1]).

The present note has been written from an attempt to attach to every real n-torus $(n \geq 2)$ a unique conformal modulus which accords with the postulates above reviewed. We set the problem simply as an immediate and natural extension of the one in the classical case of $\boldsymbol{C} = \boldsymbol{R}^2$. Our process to approach the purpose rests on the extremal length method also familiar to most of complex analysts. We collect notations employed afterwards, explain the basic objects and remark an elementary algebraic fact which plays an important rôle in the subsequent analytic considerations in § 1. In §§ 2$\sim$3 we study the foliation of vector fields generated by axes of infinitesimal ellipsoids which characterize the given diffeomorphism. §§ 4$\sim$5 deals, after certain heuristic examinations, with the extremal quasiconformal mapping between rectangular parallelepipeds which turns out an n-dimensional version of Grötzsch's *möglichst konforme Abbildung* of a rectangle onto another rectangle. We state and prove the main theorem in §§ 5$\sim$6 which asserts the existence and uniqueness of the extremal