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ON THE COMMUTATIVITY OF THE RADICAL OF THE GROUP ALGEBRA OF AN INFINITE GROUP

YASUSHI NINOMIYA

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Throughout K will represent an algebraically closed field of characteristic $p>0$, and G a group. Let G' be the commutator subgroup of G. The Jacobson radical of the group algebra KG will be denoted by $J(KG)$. In case G is a finite group and p is odd, D.A.R. Wallace [6] proved that $J(KG)$ is commutative if and only if G is abelian or $G'P$ is a Frobenius group with complement P and kernel G', where P is a Sylow p-subgroup of G. On the other hand, when we consider the case $p=2$, by the following theorem, we may restrict our attention to the case $|P|\geqq 4$.

Theorem 1 ([5]). *Let G be a group of order $p^a m$. where $(p, m)=1$. Then $J(KG)^2=0$ if and only if $p^a=2$.*

In the previous paper [3], we obtained the following

Theorem 2. *Let $p=2$, and G a non-abelian group of order $2^a m$, where m is odd and $a\geqq 2$. Then the following conditions are equivalent:*

(1) *$J(KG)$ is commutative.*

(2) *G' is of odd order and $|P\cap P^x|\leqq 2$ for each $x\in G'P-P$.*

(3) *G' is of odd order and $C_{G'P}(s)/\langle s\rangle$ is either a 2-group or a Frobenius group with complement $P/\langle s\rangle$ for every involution s of P.*

(4) *G' is of odd order and each block of $KG'P$, except the principal block, is of defect* 1 *or* 0.

In case G is an infinite group and p is odd, D.A.R. Wallace [8] gave also a necessary and sufficient condition for $J(KG)$ to be commutative. Let G be an infinite non-abelian group. We suppose that *$J(KG)$ is non-trivial.* By [8], Theorem 1.1, if $p=2$ and $J(KG)$ is commutative, then the following three cases can arise:

(α) G' is an infinite group and $J(KG)^2=0$.

(β) G' is a finite group of odd order.

(γ) G' is a finite group of even order and the order of a Sylow 2-group P of G is not greater than 4.