Motose, K. and Ninomiya, Y.
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ON THE COMMUTATIVITY OF THE RADICAL OF THE GROUP ALGEBRA OF A FINITE GROUP

KAORU MOTOSE AND YASUSHI NINOMIYA

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Let K be an algebraically closed field of characteristic $p>0$, and G a finite group of order $p^a m$ where $(p, m)=1$ and $a>0$. We denote by $J(KG)$ the radical of the group algebra KG. In case p is odd, D.A.R. Wallace [6] proved that $J(KG)$ is commutative if and only if G is abelian or $G'P$ is a Frobenius group with complement P and kernel G', where P is a Sylow p-subgroup of G and G' the commutator subgroup of G. On the other hand, in case $p=2$, S. Koshitani [1] has recently given a necessary and sufficient condition for $J(KG)$ to be commutative. In this paper, we shall give alternative conditions for $J(KG)$ to be commutative.

If $J(KG)$ is commutative, then G is a p-nilpotent group and a Sylow p-subgroup of G is abelian ([6], Theorem 2). We may therefore restrict our attention to a p-nilpotent group. Now, we put $N=O_{p'}(G)$. For a central primitive idempotent ε of KN, we put $G_\varepsilon=\{g\in G \mid g\varepsilon g^{-1}=\varepsilon\}$. Let a_i $(i=1, 2, \cdots, s)$ be a complete residue system of $G(\text{mod } G_\varepsilon)$

$$G = G_\varepsilon a_1 \cup G_\varepsilon a_2 \cup \cdots \cup G_\varepsilon a_s\,.$$

Then K. Morita [2] proved the following:

Theorem 1. *If G is a p-nilpotent group, then $e=\sum_{i=1}^{s}\varepsilon^{a_i}$ is a central primitive idempotent of KG and KGe is isomorphic to the matrix ring $(KP_\varepsilon)_f$ of degree f over KP_ε for some f, where P_ε is a Sylow p-subgroup of G_ε.*

In what follows, for a subset S of G, we denote by $\hat{S}$ the element $\sum_{x\in S}x$ of KG. By [5], Theorem, it holds that $J(KG)^2=0$ if and only if $p^a=2$. When this is the case, $J(KG)$ is trivially commutative. Therefore we may restrict our attention to the case $p^a\geqq 3$. The following proposition contains [1], Theorem 2.

Proposition. *If G is a non-abelian group and $p^a\geqq 3$, then the following conditions are equivalent:*

(1) *$J(KG)$ is commutative.*
(2) *$(G'P)'=G'$ and $J(KG'P)$ is commutative.*
(3) (i) *G' is a p'-group, and*