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ON THE NUMBER OF LATTICE POINTS IN THE SQUARE $|x|+|y|\leqq u$ WITH A CERTAIN CONGRUENCE CONDITION

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0. Introduction. Let $a(u; p, q)$ denote the number of lattice points $(x, y)\in \boldsymbol{Z}^2$ such that (i) $|x|+|y|\leqq u$ (ii) $x+py\equiv 0 \pmod q$, where u, p, and q are given positive integers. It is easy to see that $a(u; p, q)$ is determined only by p modulo q, if q is fixed. Let p' be another positive integer. We always assume $(p, q)=(p', q)=1$ in the following, where (,) means the greatest common divisor. It is easy to see that we have $a(u; p, q)=a(u; p', q)$ for every positive integer u if $p\equiv \pm p'$ or $pp'\equiv \pm 1 \pmod q$. We will prove, in the present paper, that the converse is valid:

Theorem 1. *Suppose $a(u; p, q)=a(u; p', q)$ for every positive integer u. Then $p\equiv \pm p'$ or $pp'\equiv \pm 1$* $\pmod q$.

Our problem is related with a problem in differential geometry, and gives an answer to it. Consider a 3-dimensional lens space with fundamental group of order q. We ask whether the spectrum of the Laplacian characterizes the space as a riemannian manifold. This geometric problem can be reduced to a problem in number theory. A special case of our theorem, where q is of the form l^n or $2\cdot l^n$ (l a prime number), has been shown (cf. Ikeda-Yamamoto [3]). Now our Theorem 1 gives a complete affirmative answer to the above geometric problem (see Section 7 below).

If a lattice point (x, y) satisfies the conditions (i) and (ii), so does the point $(-x, -y)$. Denote by $b(u; p, q)$ the number of lattice points (x, y) such that (i′) $x\geqq 0$ and $x+|y|=u$ (ii) $x+py\equiv 0 \pmod q$. Then we see easily that Theorem 1 is equivalent to

Theorem 2. *Suppose $b(u; p, q)=b(u; p', q)$ for every positive integer u. Then $p\equiv \pm p'$ or $pp'\equiv \pm 1$* $\pmod q$.

We introduce rational functions $F_j(X)$ $(0\leqq j\leqq q-1)$;

$$F_j(X)=\frac{1}{(1-\zeta^j X)(1-\zeta^{pj}X)}+\frac{1}{(1-\zeta^j X)(1-\zeta^{-pj}X)},$$

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