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FACTOR RINGS OF A HEREDITARY AND QF-3 RING

Dedicated to Professor Goro Azumaya on his 60th birthday

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We have been studying many interesting properties of small submodules. W.W. Leonard [8] and M. Rayar [12] defined small modules and gave elementary properties of them. Recently, the author has studied non-small modules and given a class of rings which are concerned with non-small modules and located between QF-rings and QF-3 rings [4] and [5].

In this note we shall consider two conditions (*) and (*)* in [4] and [5] (see §1) and study a semi-primary ring whose every factor ring satisfies either (*) or (*)*. We shall show such a ring with condition (QS) (see §1) coincides with a generalized uni-serial ring of the first category in the sense of Murase [9].

1. The main theorem

Let R be a ring with identity. We always assume that R is a semi-primary ring, namely the Jacobson radical J of R is nilpotent and R/J is artinian, and every R-module is an unitary right R-module unless otherwise stated. Let M be an R-module. By $E(M)$ and $J(M)$ we denote an injective hull and the Jacobson radical of M, respectively. If M is a small submodule in $E(M)$, we say M is a *small module* [8], [12] and if M is not a small module, we say M is *non-small module* [5]. As the dual concept to the above, we define a *non-cosmall module* N as follows: there exist a projective module P and an epimorphism $f: P \to N$ such that $\ker f$ is not essential in P.

In [4] and [5] we have introduced two conditions:

(*) *Every non-small module contains a non-zero injective module.*

(*)* *Every non-cosmall module contains a non-zero projective direct summand.*

We have shown that if R satisfies either (*) or (*)*, then R is a right QF-3 ring [13] ($E(R)$ is projective by [7]) and every QF-ring satisfies both (*) and (*)*. Thus, a class of rings satisfying either (*) or (*)* is located between a class of QF-rings and one of QF-3 rings when R is a left and right artinian ring. If R is left and right artinian and eR, Re have unique composition series for every