Ōtsubo, S.
Osaka J. Math.
16 (1979), 667-673

ORDER OF LIOUVILLIAN ELEMENTS SATISFYING AN ALGEBRAIC DIFFERENTIAL EQUATION OF THE FIRST ORDER

SHŪJI ŌTSUBO

(Received September 25, 1978)

0. Introduction. Let k be an algebraically closed ordinary differential field of characteristic 0, and Ω be a universal extension of k. A finite chain of extending differential subfields $k=L_0\subset L_1\subset\cdots\subset L_n$ in Ω is called a Liouville chain over k if the following two conditions are satisfied:

(i) The field of constants of L_n is k_0, where k_0 is the field of constants of k;

(ii) For each $i(1\leqq i\leqq n)$ there exists a finite system of elements $w_1, w_2, \cdots, w_r$ of L_i which satisfies the following two conditions; either $w'_j\in L_{i-1}$ or w'_j/w_j is the derivative of an element of L_{i-1} for each j $(1\leqq j\leqq r)$, L_i is an algebraic extension of $L_{i-1}(w_1, w_2, \cdots, w_r)$ of finite degree.

Let z be an elemen of Ω. Then, z is called a liouvillian element over k if there exists a Liouville chain over k such that its end contains z. The following definition is due to Liouville [2] (cf. [8, p. 111]):

DEFINITION. A liouvillian element z over k is said to be of order m if m is the minimum of those n such that the end of a Liouville chain $L_0\subset\cdots\subset L_n$ over k contains z.

Let F be an algebraically irreducible element of the first order of the differential polynomial algebra $k\{u\}$ in a single indeterminate u over k. Suppose that z is a solution of $F=0$. Then, z is a generic point of the general solution of $F=0$ over k if and only if z is transcendental over k. Suppose that two liouvillian elements over k satisfy $F=0$ and that they are transcendental over k. Then, their orders are the same.

Theorem. *The order of a liouvillian element over k satisfying $F=0$ is at most three.*

For example, suppose that k is the algebraic closure of $k_0(x)$ with $x'=1$ and that $F=u'-\alpha u/x$, where $\alpha\in k_0$. Then, any non-trivial solution of $F=0$ is of the second order if α is not a rational number (cf. Liouville [2, pp. 94–98]).

REMARK 1. If we replace "liouvillian" by "generalized elementary" and