Sumioka, T.
Osaka J. Math.
**16** (1979), 395-403

# ON QF-3 AND 1-GORENSTEIN RINGS

TAKESHI SUMIOKA

(Received May 6, 1978)

Let $R$ be a ring with identity, $E$ the injective hull of $R_R$ and $Q$ the maximal right quotient ring of $R$. Consider the following conditions.

$A_1$: $E$ is projective.

$A_2$: $R$ has a minimal faithful right $R$-module.

$A_3$: $R$ has a faithful injective right ideal.

$A_4$: $E$ is torsionless.

$A_5$: Any finitely generated submodule of $E$ is torsionless.

$A_6$: Any finitely generated submodule of $Q$ is torsionless.

For arbitrary ring $R$, $A_2 \Rightarrow A_3 \Rightarrow A_4 \Rightarrow A_5 \Rightarrow A_6$ hold (see [14, p, 47]). On the other hand if $R$ is a left perfect ring, $A_2$, $A_3$ and $A_4$ are equivalent (see [4, Theorem 3.2], [1, Theorem 2] and [13, Proposition 3.1]). Any commutative integral domain which is not a field satisfies $A_5$, but does not satisfy $A_4$. However, Rutter [8, Corollary 3] proved that for any right or left artinian ring $R$ $A_5$ implies $A_2$. In § 2 we shall define a new ring, called a right DWA ring, which is a generalization of a right semi-artinian ring, and we shall show that for any right DWA and right perfect ring $R$ $A_5$ implies $A_3$. As a consequence, for any perfect ring $R$ $A_5$ implies $A_2$. We owe the method of the proof essentially to Rutter [8, Lemma 2] and the proofs of Colby and Rutter [1, Theorem 2] and Jans [4, Theorem 3.2]. We call a ring $R$ right QF-3 if $R$ satisfies $A_5$ and $R$ right QF-3′ if $R$ satisfies $A_6$.

By Wu, Mochizuki and Jans [16] (or Colby and Rutter [1, Theorem 1] and Kato [6, Remark, p. 236]), a characterization of right artinian rings satisfying $A_1$ (or rings satisfying $A_4$) was given. In § 3 we shall give an analogous result for right QF-3 rings. In § 4 we shall show that any right QF-3′ and right 1-Gorenstein ring is QF-3 (see § 4 for the definition of a 1-Gorenstein ring). Moreover using this result, we shall show that any noetherian right QF-3′ ring such that $E \oplus E/R$ is an injective cogenerator is right and left 1-Gorenstein.

## 1. Definitions and notations

Throughout this note we assume that $R$ is a ring with identity and all $R$-modules are unitary. We denote by $E$ the injective hull of $R_R$ and by $Q$ the