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A DIOPHANTINE EQUATION ARISING FROM TIGHT 4-DESIGNS

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Ito [1,2] and Enomoto, Ito, Noda [3] show that there exist only finitely many tight 4-designs, by proving that such a design gives rise to a unique rational integral solution of the diophantine equation

$$(2y^2-3)^2 = x^2(3x^2-2) \tag{1}$$

and then invoking a result of Mordell [4] to say that this equation has only finitely many solutions in integers x,y. A privately communicated conjecture is that (1) has only the 'obvious' solutions $(\pm x, \pm y)=(1,1), (3,3)$, with the implication that the only tight 4-designs are the Witt designs. We show here that this is indeed the case.

We are exclusively interested in integral points on the curve (1), which is a lightly disguised elliptic curve; standard arguments show that the group of rational points has one generator of infinite order which may be taken to be (3,3).

Suppose now that x, y are integers satisfying (1). Then there is an integer w with

$$\begin{aligned} 3x^2-2 &= w^2 \\ 2y^2-3 &= wx\,. \end{aligned} \tag{2}$$

Clearly x,w,y are odd. Following Cassels [5] we write (2), in virtue of the identity $w^2-3x^2+2wx\sqrt{-3}=(w+x\sqrt{-3})^2$, in the form

$$\left(\frac{w+x\sqrt{-3}}{2}\right)^2-y^2\sqrt{-3} = \frac{-1-3\sqrt{-3}}{2} \tag{3}$$

We now work in the algebraic number field $Q(\theta)$ where $\theta^2=\sqrt{-3}$. It is easy to check that the ring of integers of $Q(\theta)$ has $\boldsymbol{Z}$-basis $\left\{1, \theta, \frac{1+\theta^2}{2}, \frac{\theta+\theta^3}{2}\right\}$, that the class-number is 1, and that the group of units is generated by $\{-\omega, \omega+\theta\}$ where $\omega=\frac{-1-\theta^2}{2}$ is a cube root of unity. The relative norm to $Q(\sqrt{-3})$ of