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Osaka J. Math.
16 (1979), 349-352

A REMARK ON SIMPLE SYMMETRIC SETS

Dedicated to Professor Yoshikazu Nakai on his 60th birthday

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(Received June 22, 1978)

Nobusawa [1] has shown that if A is a simple symmetric set then the group of displacements $H(A)$ is almost simple. The purpose of this note is to prove the converse. For the completeness we shall restate the result of Nobusawa in a slightly extended way.

A symmetric set is a set A carrying a binary operation $a\circ b$ which satisfies the following identical relations:

(1) $a\circ a=a$.
(2) $(x\circ a)\circ a=x$.
(3) $(x\circ y)\circ a=(x\circ a)\circ(y\circ a)$.

The mapping $\iota(a)$: $A\rightarrow A$ defined by $x^{\iota(a)}=x\circ a$ is an automorphism of A by (3) and we have the following:

(4) $a^{\iota(a)}=a$.
(5) $\iota(a)^2=1$.
(6) For any automorphism σ of A

$$\sigma^{-1}\iota(a)\sigma=\iota(a^{\sigma}).$$

Particularly we have

(7) $\iota(b)^{-1}\iota(a)\iota(b)=\iota(a^{\iota(b)})=\iota(a\circ b)$.

The group $G(A)$ is the subgroup of Aut A (the automorphism group of A) generated by $\iota(A)=\{\iota(a)\,|\,a\in A\}$. The group $H(A)$ is the subgroup of $G(A)$ generated by $\{\iota(a)\iota(b)\,|\,a, b\in A\}$ and is called the group of displacements. Then $|G(A): H(A)|\leqq 2$ and $H(A)=\langle\iota(e)\iota(a)\,|\,a\in A\rangle$ for a fixed element e of A.

The set $\iota(A)$ is a collection of conjugate classes of involutions in $G(A)$, and is a symmetric set with the binary operation $\iota(a)\circ\iota(b)=\iota(b)^{-1}\iota(a)\iota(b)$. The mapping ι: $A\rightarrow\iota(A)$ is an epimorphism, and if ι is an isomorphism then A is called *effective*.

If $G(A)$ acts transitively (or primitively) on A, then we call A *transitive* (or *primitive*). Note that A is transitive if and only if $H(A)$ is transitive on A, and if A is transitive then $\iota(A)$ is a conjugate class of involutions in $G(A)$.