Hirashima, Y.
Osaka J. Math.
16 (1979), 219–231

A NOTE ON COHOMOLOGY WITH LOCAL COEFFICIENTS

Yasumasa HIRASHIMA

(Received November 22, 1977)
(Revised March 11, 1978)

For the ordinary cohomology groups of a good space X we have the homotopy classification theorem, $H^q(X; G)\cong[X, K(G, q)]$. It has been discussed by many authors (Olum [9], Gitler [5], Siegel [12] and McClendon [8]) that a similar theorem is valid for the cohomology with local coefficients. In this paper we give an elementary reasonable proof of the classfication theorem by means of direct generalizations of the results in May [6] and Cartan [1].

We frequently use the notations introduced in [6] without notice.

1. Definitions and main theorems

Let S denote the category of simplicial sets, in which we shall work. Let π be an abstract group, and let G be an abelian group. We fix a group homomorphism ϕ: $\pi\rightarrow AutG$, where $AutG$ is the automorphism group of G. Let $(X, Y; \tau)$ be a simplicial pair (X, Y) (Y may be empty) with a twisting function (see [6]) τ: $X\rightarrow\pi$. We define the group of cochains $C^n_\phi(X, Y; \tau)$ to be

$$\{f: X_n\rightarrow G\,|\,f(x)=0 \text{ if } x=s_iy \text{ or } x\in Y_n\}\ ,$$

the coboundary δ by

$$\delta f(x)=\tau(x)^{-1}f(\partial_0x)+\sum_{i=1}^{n+1}(-1)^if(\partial_ix),\ x\in X_{n+1}, f\in C^n_\phi(X, Y; \tau)\,.$$

$H^n_\phi(X, Y; \tau)=H^n_\phi(C^*_\phi(X, Y; \tau), \delta)$ is called the twisted cohomology group of $(X, Y; \tau)$ by ϕ.

Let L be a local system on X, i.e. a contravariant functor from the fundamental groupoid (see [4]) πX to the category of abelian groups. Suppose X is connected. We fix a vertex $x_0\in X_0$ and $u_x\in\pi X(x_0, x)$, $x\in X_0$, in particular we choose $u_{x_0}=1_{x_0}$. Then we have the twisting function $F(x_0, (u_x))$: $X\rightarrow\pi_1X$ by

$$F(x_0, (u_x))(y)=u^{-1}_{\partial_0\partial_2\cdots\partial_ny}\partial_2\cdots\partial_ny\,u_{\partial_1\partial_2\cdot\partial_ny},\ y\in X_n\ ,$$

a group homomorphism $\phi(L)$: $\pi_1X\rightarrow AutL(x_0)$ by