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LEFT-INVARIANT LORENTZ METRICS ON LIE GROUPS*)

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With J. Milnor [2] we consider a special class $\mathfrak{S}$ of solvable Lie groups. A non-commutative Lie group G belongs to $\mathfrak{S}$ if its Lie algebra $\mathfrak{g}$ has the property that $[x, y]$ is a linear combination of x and y for any elements x and y in $\mathfrak{g}$. It is shown that $\mathfrak{g}$ has this property if and only if there exist a commutative ideal $\mathfrak{u}$ of codimension 1 and an element $b\notin\mathfrak{u}$ such that $[b, x]=x$ for every $x\in\mathfrak{u}$.

Milnor has shown that if $G\in\mathfrak{S}$, then every left-invariant (positive-definite) Riemannian metric on G has negative constant sectional curvature. The simplest example is given by

$$G=\left\{\begin{bmatrix} a & b \\ 0 & 1\end{bmatrix};\ a>0, \quad a, b\in R\right\}.$$

On the other hand, Wolf [3, p. 58] showed that this group G admits a left-invariant Lorentz metric which is flat (that is, with zero sectional curvature).

Our first and main objective in this paper is to prove the following theorem.

Theorem 1. *If a Lie group G belongs to the class $\mathfrak{S}$, then*

(1) *every left-invariant Lorentz metric (of signature $(-, +, \cdots, +)$) has constant sectional curvature;*

(2) *given any arbitrary constant k, $k>0$, $k=0$, or $k>0$, one can find a left-invariant Lorentz metric on G with k as constant sectional curvature.*

Unlike the Riemannian case, the existence of a flat left-invariant Lorentz metric seems to be a more frequent phenomenon. Our second objective is to prove

Theorem 2. *Each of the following 3-dimensional Lie groups admits a flat left-invariant Lorentz metric:*

(1) *$E(2)$: group of rigid motions of Euclidean 2-space;*

(2) *$E(1, 1)$: group of rigid motions of Minkowski 2-space;*

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