SPECTRA AND EIGENFORMS OF THE LAPLACIAN ON $S^n$ AND $P^n(\mathbb{C})$

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Introduction

Let $M$ be a compact Riemannian manifold. We consider the Laplace operator $\Delta$ acting on the space of differential forms on $M$. It is a strongly elliptic self-adjoint differential operator, so it has discrete eigenvalues with finite multiplicities. For a given Riemannian manifold, it may be an interesting problem to determine explicitly the spectra and eigenforms of $\Delta$ on $M$. As for the spectra and eigenfunctions of $\Delta$ acting on the space of functions, they are known for the cases where $M$ are the following manifolds; flat tori, Klein bottles [3], symmetric spaces [12] and the Hopf manifolds [1]. On the other hand, as for the spectra and eigenforms of $\Delta$ acting on the differential forms, we have known no results except for flat tori. But, E. Calabi (unpublished) and recently S. Gallot et D. Meyer [7] have computed the eigenvalues of differential forms on the standard sphere by using the harmonic polynomial forms on $\mathbb{R}^{n+1}$.

In this paper, applying the representation theory we compute the eigenvalues of $\Delta$ and determine the spaces of eigenforms as representation spaces, on the standard sphere $S^n$ and the complex projective space $P^n(\mathbb{C})$ with Fubini-Study metric. Our method is as follows: Let $M=G/K$ be a Riemannian homogeneous space with $G$ acting as transitive isometry group on $M$. Then $\Delta$ is a $G$-invariant differential operator, so its eigenspaces are $G$-modules. First, we decompose the space of differential forms on $M$ into $G$-irreducible modules. In the case where $M$ is $S^n$, $P^n(\mathbb{C})$, or more generally a symmetric space, roughly speaking $\Delta=-\text{Casimir operator on } G$. So from Freudenthal's Formula, we can compute the eigenvalues. But the first step of decomposing the space of differential forms on $M$ into $G$-irreducible modules is generally not easy. In virtue of Frobenius reciprocity law, the problem can be reduced to the following problem: For a given irreducible $G$-module, how does it decompose into irreducible $K$-modules? For this problem, a few results are known (cf. H. Boerner [3] and D. P. Zelobenko [15]), and in case $M=S^n$, we apply the known results.

As for the Laplacian $\Delta$ acting on the space of functions on $S^n$ and $P^n(\mathbb{C})$, its...