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COMPACT LOCALLY HESSIAN MANIFOLDS

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Let M be a differentiable manifold with a locally flat linear connection D. A Riemannian metric g on M is said to be *locally Hessian* if for each point $p\in M$ there exists a C^∞-function φ defined on a neighbourhood of p such that $g=D^2\varphi$. Such a pair (D, g) is called a *locally Hessian structure* on M [4].

Let M be a differentiable manifold with a locally Hessian structure (D, g). Throughout this note the local expressions for the locally Hessian metric and the related concepts will be given in terms of affine local coordinate systems with respect to D. Let v be the volume element determined by the Riemannian metric g;

$$v = F dx^1\wedge\cdots\wedge dx^n, \quad \text{where} \quad F=\sqrt{\det[g_{ij}]}\,.$$

We define the forms α and β by

$$\alpha_i = \frac{\partial \log F}{\partial x^i},$$

$$\beta_{ij} = \frac{\partial^2 \log F}{\partial x^i \partial x^j},$$

and call them the Koszul form and the canonical bilinear form respectively. These forms α and β play important roles in the study of locally Hessian manifolds [2] [3] [4] [5].

The following assertion is derived from a result of Koszul [3].

(a) *Let M be a compact connected differentiable manifold with a locally Hessian structure (D, g). If the canonical bilinear form β is positive definite on M, then the universal covering manifold of M with a locally Hessian structure induced by (D, β) is isomorphic to an open convex domain not containing any full straight line in a real affine space.*

In our viewpoint a theorem of Calabi [1] is stated as follows:

(b) *Let M be a domain in the n-dimensional real affine space and let φ be a C^∞-function on M such that $g=D^2\varphi$ is positive definite, where D is the natural flat linear connection on M (Thus $(D\ g)$ is a locally Hessian structure on M). If the Riemannian metric g on M is complete and if the Koszul form α vanishes identically*