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ON SMALL RING HOMOMORPHISMS

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The author studied the total quotient ring of a commutative ring R from the point of view of small R-submodules [2]. In this note, we shall extend those methods to a ring extension of R. Let R and R' be commutative rings and $f: R \rightarrow R'$ a ring homomorphism. If $f(R)$ is a small R-submodule of R', we say f being *small* or R being *small* in R'. In the first section, we shall give a criterion for R to be small in R' in terms of maximal ideals in R and R' and obtain fundamental properties of small homomorphisms. In the second section, we shall give a characterization of maximal ideals M by the multiplicative systems $R-M$ and small homomorphisms.

Throughout this note, we assume every ring R is a commutative ring with identity unless otherwise stated and very ring homomorphism is also unitary, i.e. $f(1)$ is the identity.

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1. Small homomorphisms

Let R be a (commutative) ring and let $M \supseteq N$ be R-modules. N is called a *small submodule in* M if it satisfies the following condition: the fact $M=N+T$ for some R-submodule T implies $T=M$. Let R' be commutative and f: $R \rightarrow R'$ a ring homomorphism. Then every R'-module may be regarded as an R-module via f. If $f(R)$ is a small R-submodule in R', we say that f is *small* or R is *small in* R'. Let A and A' be ideals in R and R', respectively. We put $f(A)R'=AR'$ and $f^{-1}(f(R)\cap A')=A'\cap R$. We shall denote the set of prime ideals by spec(R) and the set of maximal ideals by Spec(R). Then we have the induced map f_*: spec$(R') \rightarrow$ spec(R).

The following lemma is well known and the proofs are trivial.

Lemma 0. 1) *Let $X \supseteq Y \supseteq Z$ be R-modules. If Z is a small R-submodule in Y, so is in X and if Y is small in X, so is Z.* 2) *Let W be an R-module and $f: X \rightarrow W$ an R-homomorphism. If Z is small in X, $f(Z)$ is small in W.* 3) *Furthermore, if U is a small submodule in W, $Z \oplus U$ is small in $X \oplus W$.*