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# REMARKS ON DIFFERENTIAL OPERATORS ON ALGEBRAIC VARIETIES

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We will work over an algebraically closed field of characteristic zero. Let $A$ be a (commutative) $k$-algebra and $I\subset A\otimes_k A$ the 'diagonal ideal', i.e., the ideal generated by $z\otimes 1-1\otimes z$, $z\in A$. We consider $A\otimes_k A$ as an $A$-module via

$$a(x\otimes y) = ax\otimes y$$

Recall that the $A$-module of $n$-jets (over $k$), $P^n(A|k)\stackrel{df}{=\!=} A\otimes_k A/I^{n+1}$ is of finite type when $A$ is a localization of a $k$-algebra of finite type. Note that there is a splitting

$$P^n(A|k) = A\otimes_k A/I^{n+1} \rightarrow A\otimes_k A/I \approx A$$

so that $P^n(A|k)\approx I/I^{n+1}\oplus A$. We write $P_0^n(A|k)$ for $I/I^{n+1}$. The $k$-homomorphism $d_n\colon A\to P^n(A|k)$ defined by $d_n(x)=\overline{x\otimes 1-1\otimes x}$ is called the 'universal' differential operator of order $n$ on $A$.

Now suppose $A$ is a local ring obtained from a $k$-algebra of finite type with residue field $k$. We wish to pose the question:

$Q(*)$: When does $P_0^n(A|k)$ have a free direct summand for $n$ sufficiently large.

REMARK. For $A$ regular $P_0^n$ is a free $A$ module so we are interested in $A$ non-regular.

To understand the above question recall the definition of a differential operator on $A$. Say $B=k[X_1, \cdots X_s]_{(X_1,\cdots X_s)}$ and $A=B/\mathfrak{A}$. The differential operators on $B$ of order $\leqslant n$ are the obvious ones, i.e., elements of $\mathrm{Hom}_k(B. B)$

$$D = \sum_{i_1+\cdots+i_s\leqq n} b_{i_1,\cdots i_s}\frac{\partial^{i_1}}{\partial X_1^{i_1}}\cdots\frac{\partial^{i_s}}{\partial X_s^{i_s}}, \quad b_{i_1}\cdots_{i_s}\in B$$

where the $\partial/\partial X_i$'s are the standard derivations of $B$. They form a $B$-module denoted by $\mathrm{Diff}^n(B|k)$. An $n$-th order differential operator on $A$ is a $D$ as above with $D(\mathfrak{A})\subset\mathfrak{A}$. We identify two operators whose difference sends $B$ into $\mathfrak{A}$, to get an $A$ module, $\mathrm{Diff}^n(A|k)$. By the definition we get $\mathrm{Diff}^n\subset\mathrm{Diff}^{n+1}$. We can identify $\mathrm{Diff}^n$ with the $A$-dual of $P^n$ via