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ON SYMMETRIC SETS OF UNIMODULAR SYMMETRIC MATRICES

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1. Introduction

A binary system A is called a symmetric set if (1) $a\circ a=a$, (2) $(a\circ b)\circ b=a$ and (3) $(a\circ b)\circ c=(a\circ c)\circ(b\circ c)$ for elements a, b and c in A. Define a mapping S_a of A for an element a in A by $S_a(x)=x\circ a$. As in [2], [3] and [4], we denote $S_a(x)$ by xS_a. S_a is a homomorphism of A due to (3), and is an automorphism of A due to (2). Every group is a symmetric set by a definition: $a\circ b=ba^{-1}b$. A subset of a group which is closed under this operation is also a symmetric set. In this paper, we consider a symmetric set which is a subset of the group $SL_n(K)$ consisting of all unimodular symmetric matrices. We denote it by $SM_n(K)$. For a symmetric set A, we consider a subgroup of the group of automorphisms of A generated by all S_aS_b (a and b in A), and call it the group of displacements of A. We can show that the group of displacements of $SM_n(K)$ is isomorphic to $SL_n(K)/\{\pm 1\}$ if $n\geq 3$ or $n\geq 2$ when $K\neq F_3$ (Theorem 5). Also we can show that $PSM_n(K)$, which is defined in a similar way that $PSL(_nK)$ is defined, has its group of displacements isomorphic to $PSL_n(K)$ under the above condition (Theorem 6). A symmetric set A is called transitive if $A=aH$, where a is an element of A and H is the group of displacements. A subset B of A is called an ideal if $BS_a\subseteq B$ for every element a in A. For an element a in A, aH is an ideal since $aHS_x=aS_xH=aS_aS_xH=aH$ for every element x in A. Therefore, A is transitive if and only if A has no ideal other than itself. Let F_q be a finite field of q elements ($q=p^m$). We can show that $SM_n(F_q)$ is transitive if $p\neq 2$ or if n is odd, and that $SM_n(F_q)$ consists of two disjoint ideals both of which are transitive if n is even and $p=2$ (Theorem 7).

A symmetric subset B of A is called quasi-normal if $BT\cap B=B$ or ϕ for every element T of the group of displacements. When A has no proper quasi-normal symmetric subset, we say that A is simple. In [4], it was shown that if A is simple (in this case, A is transitive as noted above) then the group of displacements is either a simple group or a direct product of two isomorphic simple groups. In **4**, we show some examples of $PSM_n(F_q)$. The first example is $PSM_3(F_2)$, which is shown to be a simple symmetric set of 28 elements.