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SIMPLE SYMMETRIC SETS AND SIMPLE GROUPS

Dedicated to the memory of Dr. Taira Honda

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1. Intorduction

A binary system A is called a symmetric set if $a\circ a=a$, $(b\circ a)\circ a=b$ and $(b\circ c)\circ a=(b\circ a)\circ(c\circ a)$. These conditions imply that the right multiplication by an element a, which we denote by S_a(*i.e.*, $b\circ a=bS_a$), is an automorphism of A of order 2 leaving a fixed. Note that, if τ is an automorphism of A, then $(b\circ a)\tau=b\tau\circ a\tau$, or $S_{a\tau}=\tau^{-1}S_a\tau$. Every group is a symmetric set by $bS_a=ab^{-1}a$. Also the subset of involutions in a group is a symmetric set. For more of symmetric sets, see [3] and [4].

The group of automorphisms of A generated by all S_a ($a\in A$) is denoted by G, and the subgroup of G generated by all S_aS_b (a, $b\in A$) is denoted by H. The latter is called the group of displacements. It is easy to see that H is generated by S_aS_e (e is a fixed element and $a\in A$). H is a normal subgroup of G of index 2. A subset B of A is called a symmetric subset if it is closed under the binary multiplication. Every one-point subset is a symmetric subset, and so is A. All the other symmetric subsets are called proper symmetric subsets. A symmetric subset B is called quasi-normal if $B\tau\cap B=B$ or ϕ (the empty set) for every element τ in G. Now we define a simple symmetric set to be one which has no proper quasi-normal symmetric subset. Theorem and Corollary obtained in **2** state that if A is simple then H is either a simple group or a direct product of two simple groups which are conjugate each other in G. If moreover A is finite, then $|H|=|A|^2$ in case H is not simple. Using this fact, we can show a new proof of the simplicity of the alternating group A_n ($n\geq5$) in **3** by showing that the subset of all transpositions in S_n (the symmetric group of n letters) is a simple symmetric set. This idea is carried out in **4** to obtain examples of simple symmetric sets in vector spaces with bilinear symmetric forms over F_2, the field consisting of two elements 0 and 1. As special cases, we obtain simple symmetric sets of positive roots of type E_6, E_7 and E_8 in Lie algebra theory.