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ON RELATIVELY SEPARABLE SUBALGEBRAS

Dedicated to the memory of the Late Professor T. Honda

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We shall treat, in this paper, with a relatively separable subalgebra in a certain algebra which is introduced by Azumaya [2]. Let A be an algebra over a commutative ring R, and B an R-subalgebra of A. A can be regarded as a $B\otimes A^0$-module by the natural way, where A^0 denotes the opposite copy of A. According to Azumaya, B is called *a relatively separable subalgebra in* A if A is a left $B\otimes A^0$-projective module. It seems, however, that such a sublagebra should be called *left* relatively separable because by the symmetric manner $A\otimes B^0$-projectivity of A naturally gives another relative separability which we may call *right* relative separability. In his paper [2], Azumaya has shown that every (left) relatively separable subalgebra B in A has the property of, say, (left) relative semisimplicity, that is, every left A-module is (B, R)-projective in the sense of Hochschild's relative homological algebra. We shall study some relations between relative separability and relative semisimplicity, and also study some properties of two sided (i.e. left and right) relatively separable subalgebras. We refer Auslander-Goldman [1] and DeMeyer-Ingraham [4] for separable algebras, Hattori [8] for semisimple algebras, and Hochschild [9] for relative homological algebra.

In this paper, every ring is assumed to have the unit and every module to be unitary.

1. Let R be a commutative ring and A an R-algebra. An R-subalgebra B of A is called *left* (*right*) *relatively separable in* A if A is left $B\otimes A^0$- ($A\otimes B^0$-) projective. μ denotes the canonical mapping $B\otimes A^0\rightarrow A$; $\mu(b\otimes a^0)=ba$. μ is a $B\otimes A^0$-epimorphism. So B is left relatively separable in A if and only if the mapping μ has a $B\otimes A^0$-right inverse, and this is also equivalent to that A is $(B\otimes A^0, R)$-projective, for μ always has an R-right inverse; $a\mapsto 1\otimes a^0$. A itself is a left (or equivalently right) relatively separable subalgebra in A if and only if A is a separable R-algebra.

Proposition 1. *Let A and B be as above. Then the following conditions are equivalent.*